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Understanding the Marxian Notion of Exploitation: A Summary of the So-Called Transformation Problem Between Marxian Values and Competitive Prices

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Part One. Background Analysis

I. Introduction

IT IS WELL UNDERSTOOD that Karl Marx's model in Volume I of *Capital* (in which the "values" of goods are proportional—albeit not equal—to the labor embodied directly and indirectly in the goods) differs systematically from Marx's model in Volume III of *Capital*, in which actual competitive "prices" are relatively lowest for those goods of highest direct-labor intensity and highest for those goods of low labor intensity (or, in Marxian terminology, for those with highest "organic composition of capital"). Critics of Marxian economics have tended to regard the Volume III model as a return to conventional economic theory, and a belated, less-than-frank admission that the novel analysis of Volume I—the calculation of "equal rates of surplus value" and of "values"—was all an unnecessary and sterile muddle.¹

Neither Marx nor Engels ever conceded this; in subsequent years it has been almost a

¹ The criticism in Böhm-Bawerk [2, 1898] is merely the longest of a great number of similar critiques. Incidentally, the view cannot be sustained that Marx, only *after* he had made his mistakes in Volume I, came to realize that he had need in Volume III to abandon or qualify his doctrines of surplus-value, for already in 1865, which was prior to the 1867 publication of Volume I, he had completed the manuscripts that formed the basis for the Volume III treatment of this issue.

badge of membership to the league of Marxist theorists to subscribe to the view that the concepts of Marxian "value" are 1) philosophically fruitful, 2) sociologically and historically of interest and relevance, 3) crucial in providing insights into the nature of capitalistic *exploitation* and into the *laws of motion* of capitalist developments. In what is germane to the present survey, Marxists concerned with the technical foundations of the subject have also believed 4) that the profit rate and prices of Volume III (and hence of bourgeois economics) *must* be anchored on the total surplus deducible from Volume I's value analysis, or at least can be usefully so anchored, 5) that Marx himself showed (albeit with admittedly only approximate accuracy) how Volume I values with their microeconomic discrepancies are "transformed" into real world prices and profits, 6) that a long line of subsequent writers—including even such bourgeois economists as L. von Bortkiewicz along with Marxian analysts such as P. M. Sweezy, J. Winternitz, K. May, M. Dobb, R. Meek, *et al.*—have removed the approximations and minor inaccuracies involved in Marx's mode of transforming values into prices, so that 7) as matters now stand, Karl Marx's pioneering analysis of values and surplus-values has been finally vindicated even by

the higher mathematics of modern economic analysis.

I hope to give here a long-overdue review of this famous transformation problem. With the exception of the too-little-known 1957 contribution of Francis Seton, the existing discussions have a black magic quality to them (a 1907 algebraic procedure of Bortkiewicz is employed, but its underlying significance is never made sufficiently clear). In this age of Leontief and Sraffa there is no excuse for mystery or partisan polemics in dealing with the purely logical aspects of the problem. So that the problems which are purely logical can be cleared up to the satisfaction of Marxians and non-Marxians alike, I am abstaining here from appraising the empirical fruitfulness of the exploitation hypotheses—for this or the last century, for static or dynamic insights.²

Part One provides the stage setting to the controversy and discusses the tools needed for its understanding. Part Two, which can be read independently of the other Parts, provides a careful statement of the issues involved in the Marxian theory of exploitation of labor; I hope it will be useful to both Marxists and their critics. Part Three reviews and elucidates the various analytical issues raised by the different contributors to the literature.

I should perhaps explain in the beginning why the words “so-called transformation problem” appear in the title. As the present

² I provide below a more detailed list of references. In Dorfman, Samuelson, and Solow's 1958 book [10] there appears what is now the standard notation for systems like those of Leontief [13, 1941 and 14, 1966] and Sraffa [35, 1960]—namely, labor is treated as the zeroth input, so that the amount of labor needed to produce one unit of the j th good, coal, or the amount of the i th good, iron, needed to produce that good, are written respectively as $a_{\text{labor, coal}} = a_{0j}$ and $a_{\text{iron, coal}} = a_{ij}$. These *direct* input requirements are to be distinguished from the *total* (direct plus indirect) input requirements depicted by A_{0j} and A_{ij} —as will be made clear later. Many of the footnotes will presuppose on the part of the reader familiarity with modern economic analytics.

survey shows, better descriptive words than “the transformation problem” would be provided by “the problem of comparing and contrasting the mutually-exclusive alternatives of ‘values’ and ‘prices.’” For when you cut through the maze of algebra and come to understand what is going on, you discover that the “transformation algorithm” is precisely of the following form: “Contemplate two alternative and discordant systems. Write down one. Now transform by taking an eraser and rubbing it out. Then fill in the other one. *Voilà!* You have completed your transformation algorithm.” By this technique one can “transform” from phlogiston to entropy; from Ptolemy to Copernicus; from Newton to Einstein; from Genesis to Darwin—and, from entropy to phlogiston . . . It tells us something about the need for a systematic survey and elucidation of the transformation problem that this uncontroversial and prosaic truth is nowhere underlined in what is now a copious literature stretching over more than three-quarters of a century.³

II. The Labor Theory of Value

1. Begin with Adam Smith's⁴ “early and rude state,” where i) land is superabundant and free and ii) productive methods are of such primitive and short duration that interest and profit are somehow ignorable.⁵ Then

³ For a terse algebraic demonstration of this, see my recent article [26, 1970]. Essentially the same point is discernible in the cited Bortkiewicz and Seton works, and also in the Morishima-Seton article [21, 1961] and in Johansen's 1961 note [11].

⁴ In *The wealth of nations* [34, 1776], Book I, Ch. VI, we find: “In that early and rude state of society which precedes both the accumulation of stock and the appropriation of land, the proportions between the quantities of labour necessary for acquiring different objects seems to be the only circumstance which can afford any rule for exchanging them one for another.”

⁵ It is easier to justify ignoring land and its rent—for under primitive conditions it is easy to imagine land to be superabundant, so that it becomes a free factor and production gets carried on in a land-sated fashion. It is harder, though, to justify in a primitive community any assumption that intermediate and durable goods

certainly if it takes one hour to hunt a deer and two hours of equally simple labor to hunt a beaver, the exchange ratio must end up with deer half as dear as beaver. The net national product represented as a flow of output and as a flow of factor income could take the simple form

$$NNP = 1 \cdot \text{deer} + 2 \cdot \text{beaver} \\ \equiv 100 \text{ percent of wage income}$$

This result, that $P_{\text{beaver}}/P_{\text{deer}}=2$ and $W/P_{\text{beaver}}=1/2$ per hour, $W/P_{\text{deer}}=1$ per hour, not only agrees with the “undiluted labor theory of value.” It is also, under the postulated circumstances, the correct general equilibrium outcome according to Walras and Böhm-Bawerk! So long as deer and beaver production are both positive, the price ratio and the equilibrium ratio of marginal utilities for every consumer of the two goods will be predictable solely from the direct labor coefficients $a_{01}=1$, $a_{02}=2$.⁶ The result also agrees both with Volume I’s analysis of Marxian values and Marx’s Volume III analysis of prices.

2. We can complicate the scenario a little without altering fundamentals. Suppose coats made from beaver are the relevant final good, along with fresh-eaten venison. All of the labor in the deer industry is direct labor (“live” labor). But suppose the two hours needed to hunt a beaver must be supplemented in a second period by one

further hour of sewing provided by labor that is equally simple, equally untrained, equally pleasant or unpleasant. Then a beaver coat has three hours of total labor in it: one hour of direct (live) labor and two hours of indirect (“dead”) labor. With land superabundant and time ignorable, Smith, Ricardo, and any believer in a labor theory of value will agree that

$$P_{\text{beaver coat}}/P_{\text{deer}} = (2 + 1)/1 = 3 \\ = A_{02}/A_{01} > a_{02}/a_{01} \\ = 2/1$$

i.e., it is total embodied labor (direct and indirect, as summarized by A_{0j}) and not merely embodied direct labor (as summarized by a_{0j}) which determines the j^{th} good’s price in the undiluted-labor-theory model.

The reader can verify that the problem is not essentially more complicated if we additionally assume that, say, four units of deer are needed as bait to hunt a unit of beaver. This merely adds further to the labor embodied indirectly in a beaver coat: now we have $A_{02}=4+2+1=7$ hours in all. Most generally, if we stick with the simple Austrian “recursive” pattern, in which every final product can ultimately be decomposed into the labor at “earlier” stages, the embodied-labor ratios A_{02}/A_{01} can always be calculated by a *finite* multiplier chain.⁷

3. Leontief, and Marx before him (in Volume II’s discussion of models of simple reproduction), goes beyond this recursive Austrian case to recognize that any output may also be needed as an input: to produce corn requires coal; to produce coal requires corn; to produce corn requires corn—indirectly or, as in the case of seed, directly. Recall these input-output coefficients are written as a , in contrast to a direct labor re-

are in such superabundant supply that time-sated productive methods are feasible. Rude economies are not low or zero interest economies; they tend to be high interest states, in which short-time methods are used precisely because of the very “scarcity of time,” incident to the serious problem of living from harvest to next harvest and from random scarcity of game and crops to random abundance. As will become clear, we must therefore make some heroic abstractions to give Smith the rope he needs for his argument—*e.g.*, an assumption of instantaneous production of deer and beaver.

⁶ For an elementary textbook discussion of the simple labor theory of value, see Samuelson [27], pp. 26–28, 712–26 of the 1970 (8th) edition, or pp. 27–29, 708–22 of the 1967 (7th) edition.

⁷ In technical terms, the Austrian model can be written with $a_{ij}=0$ when $i \geq j$, so that $I-a$ has all diagonal and below-diagonal elements zero, with the consequence that $[I-a]^{-1}=I+a+a^2+\dots+a^{n-1}$, a finite series by virtue of the fact that $a^n=0=a^{n+t}$, where n is the number of goods in the system.

quirement already written as a_0 . (E.g., $a_{\text{coal, corn}} = a_{35}$, $a_{\text{coal, coal}} = a_{33}$, $a_{\text{labor, coal}} = a_{03}$ —the first subscript denoting input, the second denoting output, and labor being so to speak the zeroth industry.)

The general notion of input-output can be quickly reviewed by positing that along with labor it takes deer itself to hunt deer (as bait, decoy, or seed). The $a_{\text{deer, deer}}$ or a coefficient must be a fraction, the (Hawkins-Simon) condition⁸ for the system to be capable of producing net final deer meat. Suppose it takes $a=3/4$ deer to produce a deer, and $a_{01}=1$ labor also. Obviously, in the steady state, you must produce four deer gross in order to have one deer net left for consumption. So what must be embodied total labor hours in a deer? Obviously, $A_{01} = a_{01}4 = 4$. To get I deer net, you must produce gross $[I-3/4]^{-1}$, or generally $[I-a]^{-1}$. (Example: if bait requires $1/2$, we need $2 = 1/(1-1/2)$; if $2/3$, we need $3 = 1/(1-2/3)$; if $1/3$, we need $1/(1-1/3) = 3/2$, which does leave one left over net.)

This total A_0 embodied labor coefficient can always be written as

$$(1) \quad A_0 = a_0[I - a]^{-1}$$

The accuracy of this result can be verified by going back in time to add up the dead labor needed at *all* the previous stages. Thus, for $a=3/4$, we need $a_0 \cdot 1$ for live labor to produce the 1 of final deer. How much direct labor was needed one period back to produce the $3/4$ deer of bait? Clearly $a_0 3/4$. And two periods back, $a_0(3/4)^2$. And so forth for $a_0(3/4)^3, \dots, a_0(3/4)^n$, in a never-ending chain. In all, we find

⁸ This condition, which was in fact discovered in connection with discussion of dynamic Marxian systems, is discussed in Dorfman-Samuelson-Solow [10, 1958] and boils down to the requirement that it must not take to produce one unit of any good, directly and indirectly, more than one unit of itself as input. For an indecomposable non-negative matrix a , the Hawkins-Simon condition is equivalent to the requirement that $[I-a]X=C$ has a positive solution X for some $C>0$, or that $\det [I-a]>0$.

$$\begin{aligned} A_0 &= a_0 + a_0 3/4 + a_0(3/4)^2 + a_0(3/4)^3 + \dots \\ &= a_0 [I + 3/4 + 9/16 + 27/64 + \dots] \\ &= a_0 \frac{I}{I - 3/4} = a_0 4 = 4 \end{aligned}$$

We have here used the high-school algebra formula for a convergent infinite-geometric progression, $I+h+h^2+\dots=(I-h)^{-1}$, where h is less than one in absolute value. The numerals “ I ” and “ 1 ” are used interchangeably here, to prepare for the matrix case where “ I ” is given a special meaning as “unity.”

4. To prepare for Marx and Leontief’s later arithmetic, suppose a taxing government rears its head in Smith’s early and rude state of the undiluted labor theory of value. It can levy a *turnover* tax, of say, $r=10$ percent, on all transactions. Or, alternatively, a value-added tax on the labor payments of say, $s=50$ percent. What is the “incidence” upon competitive price charged for deer, when that price is expressed in the same old wage units (e.g., hours)? Now $A_0=4$ is definitely too small.

Let us calculate each case separately: for the turnover tax, which will correspond to Volume III’s profit-price model; and for the value-added tax, which will correspond to Volume I’s surplus value model. I begin with the turnover tax of $r=.10$.

In the simplest case, where two direct hours produce one beaver and one direct hour produces one deer, the price in both industries is marked up by ten percent. The new exchange ratio is clearly given by $2(1.1)/1(1.1) = a_{02}(1.1)/a_{01}(1.1)$. Exchange ratios are still the same as those given by embodied labor contents, because the tax happens to cancel out.

In the case where a beaver coat requires two units of labor to hunt the beaver and one more unit of labor of sewing, the matter is not quite so simple. The price of a coat includes the tax once on the one unit of direct sewing labor; it also includes a pyramided

markup on the one unit of raw beaver input, which already has a tax in it from the two units of labor of earlier stage. Now the cost of a coat is $[2(1.1) + 1](1.1) = 2(1.1)^2 + 1(1.1) = 3.52$, and its price ratio to simple deer price is

$$\begin{aligned} & [2(1.1)^2 + 1(1.1)]/1(1.1) \\ &= [2(1.1) + 1]/1 = 3.2/1 \\ &= A_{02}/A_{01} > a_{02}/a_{01} = [2 + 1]/1 \end{aligned}$$

Summary. Because industries are unequal in their relative-direct-labor intensities (organic compositions of capital), a turnover tax pyramids or compounds differently in the various industries, leading to exchange ratios that deviate from those given by embodied labor hours. Those with relatively most labor “dated far back” rise most in relative price.

Thus formula (1) clearly had need to be modified. A ten percent turnover tax must be paid at every stage and applies to non-labor costs as well as direct labor costs: its effects on prices are exactly those of supposing that every input requirement, a and a_0 , is artificially stepped up by 1.1. So equation (1) will become in the case of a turnover tax of $1+r$,

$$(2) \quad A_0(r) = a_0(1+r)[I - a(1+r)]^{-1} > A_0(0) = A_0 \text{ of (1) if } r > 0.$$

The reader can verify this by adding the pyramided tax at all the stages or turnovers, namely

$$\begin{aligned} & a_0(1.1) + a_0(1.1)\{a(1.1)\} + a_0(1.1)\{a(1.1)\}^2 \\ & + \dots = a_0(1.1)[I + \{a(1.1)\} \\ & \quad + \{a(1.1)\}^2 + \dots] \\ & = a_0(1.1)[I - a(1.1)]^{-1} \end{aligned}$$

This demonstrates a further obvious truth: raising r , as from .1 to .2, must add to cost at every stage, and hence in toto: so $A_0(r)$ is an ever-rising function of r , with $A_0'(r) > 0$.

Indeed, r must not get too large: if it takes 3/4 deer as bait for one deer, r must stay below $33\frac{1}{3}$ percent. This is because no price could recoup a larger tax rate, since for $r = \frac{1}{3}$,

$$\begin{aligned} A_0(1/3) &= a_0(1+1/3) + 3/4 A_0(1/3)(1+1/3) \\ &= a_0 4/3 + (3/4)(4/3) A_0(1/3) \end{aligned}$$

showing that nothing would be left over for wages at so high a tax!

Now look at the value-added tax. This case is much simpler. The tax does not pyramid. It is paid once on direct labor at each of the many stages. If the tax is $s = .3$, then the effect is exactly as if you must pay for 1.3 hours of labor where before you paid for 1. Thus each a_0 becomes $a_0(1.3)$, but the input-output coefficients for raw materials, a , are left quite unchanged. So (1) is modified for the value-added case to read⁹

$$(3) \quad A_0(0)(1+s) = a_0(1+s)[I - a]^{-1}$$

Summary. A value-added tax on labor leaves all price ratios the same as embodied labor ratios, marking up all prices by the same percentage.

In concluding the tax arithmetic, note that one could have a “transformation” problem or a “contrast and compare” problem between the two tax regimes. For all r s, there is one spread of results; for all s s, there is a different spread of results. It is not clear how one would want to pair off a particular r^* and s^* for comparison. But, if one were a libertarian who regarded the government as a voracious octopus that takes resources away from people, one could postulate a theory of government exploitation in which, whether by a turnover tax r^* or a value-added tax s^* , taxation leaves the private worker with the same minimum-subsistence real wage. In

⁹ The reader not well-versed in algebra is reminded that A_0 is short for $[A_{01}, A_{02}, \dots]$ and that (3) is a terse way of saying: “price of any good when the rate of surplus value is positive is equal to its embodied labor content multiplied by one-plus-the-rate-of-surplus-value.”

either regime the worker, so to speak, works six hours of the twelve-hour day for himself and six hours for the exploiting octopus. If the needed minimum amount of subsistence is m (say $m=1/10$ deer per labor unit), we can solve for the alternative r^* or s^* , respectively by $W/P=m=1/10$, thus

$$A_0(r)m = 1 \quad \text{or} \quad A_0(0)(1+s)m = 1$$

Where it takes $3/4$ deer as bait and one direct hunting hour and $m=1/10$, we find

$$\begin{aligned} & (1+r)[I-3/4(1+r)]^{-1} 1/10 \\ &= \frac{1+r}{10[1-3/4(1+r)]} = \frac{4(1+r)}{40-30(1+r)} = 1 \\ & 1+r^* = \frac{40}{34} = 1+6/34 \end{aligned}$$

So the turnover tax rate is $6/34$ or about $17\frac{1}{2}$ percent. Alternatively

$$\begin{aligned} (1+s)[I-3/4]^{-1} 1/10 &= (1+s)(4) 1/10 = 1 \\ 1+s^* &= \frac{10}{4} = 1+1.5 \end{aligned}$$

So the value-added tax rate is one-hundred-fifty percent. Hence $(r^*, s^*) = (6/34, 1.5)$ is the relevant pairing. Naturally $s^* > r^*$, since the turnover tax, being compounded so many times, must be at a lower rate than the only-once applied value-added tax if the same real wage is to ensue.

III. Shortcomings of the Labor Theory of Value

1. Adam Smith lingered in his "early and rude state" with its undiluted labor theory for only a page. Turn the page and Eden is left behind. Now land is scarce; rent is charged for it; deer and beaver now have exchange prices that include land-rent, and except in the singular case of goods that happen to have exactly the same labor-land intensities, price ratios forever depart from embodied labor contents.

How did the Ricardians miss this elementary fact? Most of them, most of the

time, thought that if Ricardo took his analysis out to the "external margin," where "no-rent land" was used, that he could "get rid of the complication of land." Out there, deer and beaver exchange at their labor requirements. But this is trivial nonsense; one doubts that clever David Ricardo could himself ever have been long fooled by it. For let society's tastes change from land-extensive deer hunting to labor-intensive beaver hunting: then Ricardo's hope to separate the important question of income distribution from the complicated problem of demand pricing is doomed. For now the external margin just worth cultivating is *changed*, and the *new* embodied labor ratios have to be solved for by Walrasian conditions of the type Ricardo hoped to be able to ignore.¹⁰

2. There is a second limitation on the labor theory of value: people are not all alike. Ricardo and Marx hoped to evade this difficulty by redefining new units of labor power. If men are one-third as productive as women, use an hour of male labor as the lowest common denominator and then dub each female labor as being three honorary male units. In terms of the new efficiency units, $1L_1+3L_2=L$, carry on with the labor theory of value.

This is fine—if it works. Remember, Marx believed that many of the differences between highly-skilled and unskilled labor rested on the differences in past training, which was produced by earlier teaching-time labor. (When one must deal with educational labor that is still bearing fruit 40 years later, Smith's postulate that time is ignorable becomes questionable. But one objection at a time; the effect of profit and interest will soon secure its attention.)

However, the efficiency-unit device will work empirically at best only as an approximation. Natural differences show a Gaussian-like spread. "A man's a man, for all of that"

¹⁰ See the cited elementary discussion [27, Samuelson, 1970] or the more complete demonstration in the seventh edition, pp. 8, 28.

is a proper legal dictum. But a woman is not a man, and men are not at any age homozygous twins. Thus, let women be three times as efficient in beaver production and two times as efficient in deer production. How do we get our new quantum of “socially-necessary labor”? By $1L_1+3L_2$? By $1L_1+2L_2$? By $1L_1+2\frac{1}{2}L_2$? All are wrong. Given the new data about female productivities along with the original knowledge that it takes twice the male labor for a beaver as for a deer, what predictions about exchange ratios can we now obtain from the labor theory of value?

The answers are, on reflection, clear. Without the Walrasian conditions of full demand equilibrium, which Ricardo wished to avoid in dealing with income distribution, little progress is possible. The beaver/deer exchange ratio can range anywhere from $4/3$ to $2/1$ depending upon whether tastes are strong for deer or for beaver. Attempting to apply a simple labor theory would result in wasteful neglect of comparative advantage (in which no woman should be producing deer while any man is producing beaver, etc.). Indeed, to understand the statics and dynamics of men-women distributive shares requires use rather than neglect of the tools of bourgeois economics (*i.e.*, of simple general equilibrium pricing).

3. An exercise in overkill of the labor theory of value has little point at this date. But, of course, the most common objection to it comes from the consideration of time. Smith’s early and rude state, although anthropologically rubbish, was logically irreproachable in its assumption of superabundant land and zero rent. Its notion that time can be ignored—that “capital” is superabundant and production is “time-saturated” under primitive conditions—was always suspect. So it is well that Smith, after turning the page of labor theory of value, does include interest or profit in competitive price along with labor wages and land rent.

Ricardo lingered longer over the labor

theory—too long his critics thought—but from the beginning he admitted that shrimp picked up on the shore, in comparison with ancient oaks or aged wine, would *not* exchange in accordance with respective embodied labor contents. It is a sad reflection on the decadence of literary economics that so much printer’s ink has been wasted on the sterile and ambiguous question of whether Ricardo had or didn’t have a labor theory of value, or a 93 percent labor theory, or

In the real world, of 1776, 1817, or 1970, time was money and interest (or profit, they are the same thing when uncertainty is ignorable) rates were not zero. Interest will compound as a cost exactly like the turnover tax of my equation (2).¹¹ So the bourgeois economics that Marx inherited at midcentury did expect competitive price ratios to differ from embodied labor contents—just as my $A_{02}(r)/A_{01}(r)$ of (2) differs from $A_{02}/A_{01} = A_{02}(0)/A_{01}(0)$ of (1). Only by stepping up past-dated labor by the compounding factor of interest, by $a_0(1+r)$ and $a(1+r)$ and by

$$\begin{aligned} &a_0(1+r) + a_0(1+r)\{a(1+r)\} \\ &\quad + a_0(1+r)\{a(1+r)\}^2 \\ &\quad + \dots \text{etc.}, \end{aligned}$$

can one calculate actual competitive costs and prices of production.

—Karl Marx, in the posthumous 1894 Volume III of *Capital*, did concur in these arithmetical facts. But his route was a more Hegelian one—of first reaction in the form of Volume I’s analysis of “values” along the line of my Equation (3)’s value-added tax arithmetic, and then Volume III’s later synthesis by means of the so-called “transformation process.”

IV. A Pre-Marx Subsistence-Wage Model

To understand this devious path, let us now recall that the classical economists regarded labor—along with deer, or beaver,

¹¹ See Sraffa [35, 1960] or Samuelson [26, 1970 and 29, 1959].

or velvets—as also subject to a cost of production. The Malthus theory of population is a well known instance. Ricardo's theory is in its essentials identical with that of Malthus. With modern von Neumann¹² methods we can easily understand the logic and the biological linkages of the Malthus model. Although Marx admired Ricardo, he loathed Malthus as a reactionary and even plagiarist. To the extent that Marx insists on rejecting Malthusian models, we are left with a harder task of understanding just how *his* von Neumann linkages are envisaged to operate. My task here is not that of perhaps discovering that the Emperor wears no clothes and that perhaps Marx's hypothesis of a minimum-subsistence exploitative wage is not well determined by efficacious linkages. My task is to elucidate the logic of his model, on the basis of acceptance of its basic postulates and axioms.

1. Let us return to Smith's rude state: two units of labor produce a beaver, one unit of labor produces a deer. Suppose minimum daily sustenance requires $m = \text{one deer}$. Then nothing is left over for luxury consumption of beaver. And nothing is left over for a possible positive profit rate or for any taxation whether of value-added or turnover type. Actually, if a tax were imposed the population would die out, just as it would if subsistence were at $m > 1$.

But suppose, perhaps because of an invention that makes venison more digestible, the minimum needed wage drops below 1. With $m < 1$, we have a contradiction: deer and labor cannot *both* be at their costs of production in terms of each other. Why not? Because the following two equations have no consistent solution:

$$(4) \quad \begin{aligned} W/P_{\text{deer}} &= m < 1 = a_{01}^{-1} \\ &= 1/(P_{\text{deer}}/W) = W/P_{\text{deer}} \end{aligned}$$

If an obliging exploiting government came

¹² There is a voluminous modern literature on the 1931 model of von Neumann [22, 1945].

along with a tax of either $1+s^*=1/m$ or $1+r^*=1/m$, the "contradiction" would disappear. In the Marx scenario, an obliging acquisitive capitalist provides the function of appropriating the surplus, $1 - m$. He does this by commanding a positive rate of profit, $1+r^*=1/m > 1$, the exploitative rate of profit that leaves the real wage at the minimum subsistence level.

But actually this hypothesis is absurd in the context of Smith's early and rude state. What hold does the capitalist have there on the worker? What bargain can he strike? What that is useful can the employer withhold from the rude worker, who hunts where he pleases on superabundant acres and is free to eat his kill on the spot? Obviously, a vulgar Marxian is wrong to resolve (4)'s contradiction by positing $1+r^*=1/m$. And Smith himself knew better. He knew that *cost affects price only by its effect on supply*. If the real wage exceeds m of subsistence, that means population will grow. And, until Marx accepts Malthus' law of diminishing returns (to men become so numerous that land becomes crowded and no longer free), wages stay high above subsistence and population grows forever in a Neumann-Malthus golden age. Symbolically, (4) is replaced by

$$(5) \quad \begin{aligned} &\frac{1}{\text{population}} \frac{d(\text{population})}{dt} \\ &= \text{a rising function of } (a_{01}^{-1} - m) \\ &= k(W/P_{\text{deer}} - m) > 0, \text{ say,} \end{aligned}$$

where $k > 0$ and the implied solution is one of exponential growth like $e^{\lambda t}$, $\lambda > 0$.¹³

Summary. In Smith's rudest state there is no tendency whatsoever for the real wage to fall to labor's minimum cost of subsistence. Instead the number of units of labor grows exponentially. At best, the Marx formula $1+r^*=1/m$ would

¹³ Recall my *Economics* [27, Samuelson, 1970, 8th ed., p. 713, or the 7th ed., p. 708].

give us the rate of profit that owners of *slave* power, of robots producible instantaneously out of deer, could earn on their assets. Free labor is another matter.

2. To give the crude exploitation theory of wages a better run for its money, let us alter some of Smith’s strong assumptions. Now suppose subsistence consists solely of 1/3 of a new beaver coat needed each period and of nothing else. Now there is no room for deer production, for profit, or for taxation. To produce one new beaver coat, it will be recalled from our earlier example, requires two units of labor in hunting at a first stage; then in a second stage one more unit of labor in sewing—or three units of labor in all. Therefore,¹⁴ man must again work all the twelve-hour day for his minimum subsistence: hunting and sewing in the steady state. Now the costs of production, in terms of each other—of men and beaver coats, happen to be consistent, namely

$$\begin{aligned} W/P &= 1/A_0 = 1/(2 + 1) = 1/3 \\ W/P &= m = 1/3 = A_0^{-1} \\ &= 1/(2 + 1) \\ &= 1/(P/W)(1 + 0) \end{aligned}$$

Now, however, *capital goods* in the form of slain beaver are seen to be needed as raw material in the steady state. If the needed final goods are to be produced, sewers must be supplied with goods-in-process to work with.

Since this is a subsistence economy with nary a surplus, it is not clear how the synchronized state ever got off the ground and got itself started. Who went without his needed m at an earlier date? No obvious answer is forthcoming. So let us now leave

¹⁴ To relate this model more simply to Marx’s image of the exploited worker as toiling so many hours of the day for himself and the remaining working hours for the exploiting employer, my earlier assumptions of *hourly* inputs has been transmuted here to *daily* inputs—for which the reader’s indulgence is asked.

the subsistence economy for a surplus economy.

Imagine that an invention has reduced the minimum subsistence level, or cost-of-production level, to $m = 1/3.84 < 1/3 = 1/A_0$. Then, as back in (4), we have inconsistent costs of production for labor and wage goods in terms of each other. A sophisticated Marxist, trying to express his theory in terms a bourgeois economist (say, David Ricardo) could understand, would be tempted to write down

$$\begin{aligned} (6) \quad P_{\text{coat}} &= W2(1 + r)^2 + W1(1 + r) \\ &= WA_0(r), \text{ cost of production} \end{aligned}$$

$W/P_{\text{coat}} = m = 1/4.08$, labor-power cost of production

$$\begin{aligned} 4.08 &= 2(1 + r)^2 + 1(1 + r) \\ 1 + r^* &= -\frac{1}{4} + \frac{\sqrt{1 + 4(2)4.08}}{4} \\ &= \frac{-1 + \sqrt{33.64}}{4} = \frac{-1 + 5.8}{4} \\ &= 1.2 \end{aligned}$$

Thus, twenty percent is the exploitation theory’s equilibrium rate of profit, r^* , at which the real wage is down to the minimum of subsistence.

3. But is this a correct behavior equation? Maybe yes, maybe no. At least the theory now has a logical chance. If workers do not save—do not “abstain,” do not “wait”—they will be unable to provide raw materials needed for their labor to work with. If capitalists own raw beaver, they can now strike a bargain with the workers and capture some of the producible surplus, $1/A_0 - m$ or $1/m - A_0$. But how much can capitalists get? And by what methods? Let us see.

Begin with zero profit. Immediately after the invention, if r stays zero, the workers get a wage above subsistence. In Malthus-Ricardo fashion, population grows. But now beaver-raw-material will be in short supply.

How much can capitalists capture in a competitive market? Perhaps as much as $r^* = .2$. Perhaps less. Perhaps more as workers starve and begin to die out.

What steady-state golden-age permanent equilibrium is possible? That depends on capitalists' propensity to save out of each profit level r and on how much raw-material capital is needed for the production process. I shall not, in discussing 1776 or 1867 models, write down the modern Harrod-Solow-Kalecki identities.¹⁵ But I can summarize what has to be the proper empirical outcome:

Summary. The equilibrium profit rate will be between zero and the exploitation rate r^* , at a level just large enough to coax out the balanced capital formation (of beaver raw material) needed for the growing labor force to work with. The work force grows because the real wage exceeds the minimum cost of subsistence and of reproduction of labor power. Any increase in capitalists' propensity to save out of each profit rate will raise the real wage and the system's natural rate of growth, and will lower the equilibrium profit rate. If we superimpose continuing technological change on the system, real wages under developing capitalism can be presumed to rise—slowly or rapidly depending upon the nature of the innovations and the underlying biological and thrift propensities.

We are left with a model that could as legitimately be claimed by Nassau Senior as by Karl Marx or Joan Robinson! Thus suppose the bourgeois family lives forever. Suppose it acts as if it had a Pigouvian rate of subjective time preference for present over future utilities of exactly $\rho = 6$ percent. Then

¹⁵ Refer to J. Robinson [24, 1956] or N. Kaldor [12, 1956].

¹⁶ Discussion and summary of optimal control theory, as discussed by Ramsey, Uzawa, Morishima, Hicks, Shell, Cass, Samuelson and Bolow, Arrow, Kurz, Mirrlees, and many others is given in Burmeister and Dobell [7, 1970], a convenient reference work.

the equilibrium Harrod identities will *adjust themselves*, as Ramsey¹⁶ analysis proves, to $.06 = \rho < r^\infty < r^* = .20$.

4. Writers on the transformation problem have accepted, generally uncritically, the exploitation theory of profit and wages. So I must not supersede it here. Therefore, from now on, let us stipulate that labor reproduces itself mightily at the slightest rise in real wage and that capitalists are grudging savers, with the result that r^∞ is always up near its maximum r^* rate.

What this section has established is that the exploitation model can be couched in bourgeois terms free of the terminological innovations of Marx's Volume I. But let us note that the spirit of the model can also be attained by Marxian concepts of surplus value. Our comparison of the turnover and value-added taxes prepares us to understand this.

Thus, a turnover tax of 20 percent with the exploiting state providing steady-state reproduction of capital goods and wasting the surplus could achieve the described minimum-subsistence equilibrium. But a value-added tax on labor could also accomplish this. We now tax at the rate s the two hunting labor units and the one labor sewing unit, and solve the equivalent of (3)

$$\begin{aligned} 1/m &= 4.08 = 2(1 + s) + 1(1 + s) \\ (7) \quad &= 3(1 + s) = A_0(1 + s) \\ s^* &= 1.08/3 = .36 \end{aligned}$$

The needed value-added tax is 36 percent. Or as Marx would say, the "uniform rate of surplus value or labor exploitation" (applied only to the *live* labor of each stage!) is 36 percent. Of every 12 full-hour day, under *either* the Volume III profit scenario or the Volume I surplus-value scenario, the worker works 100/136 for himself and the other 36/136 of the time for the government or for the capitalists' exaction. (What is different in the alternative regimes is the exchange ratio of directly-produced beaver and indirectly-

produced coats, namely $3/1$ versus $3.4/1 = [2(1.2) + 1]/1$.)

V. Graphical Synthesis¹⁷

Figure 1, which uses numerical data from a table to come in Part Two (namely Table 2), can summarize the exploitation theory of wages and interest. The production-possibility frontier of net final goods, producible with fixed labor of say, 1, is given by *ABZ*, with slope determined by embodied labor coefficients A_{02}/A_{01} . The minimum subsistence cost-of-production level is shown by the *m'mm'* contour, whose corner at *m* specifies the market basket of needed wage goods.

There are three distinct ways of getting to *m*: 1) by direct planning, rationing, and command; 2) by a competitive profit rate *r* (corresponding to a turnover tax); 3) by a postulated equal rate of surplus value *s* (involving a markup on direct labor costs alone, on Marx's "variable capital v_j " exclusive of his "constant capital c_j ", and corresponding to a value-added tax). The first of these methods requires no further explanation.

In the second regime, charging a positive profit, $r > 0$, will push the budget-equation for the representative worker inward from *ABZ*, as the elements of $A_0(r)$ are marked up over $A_0(0)$. The inward shift will generally not be parallel, tending rather to steepen in recognition of the fact that the good with relatively less direct labor will have more of a markup from the compounded interest on its earlier-dated labor. These steepening lines will cover the chart and there will be a unique profit rate, r^* , that brings the worker's budget constraint down to *amz*, the steep line through *m*.

In the third regime, a positive rate of surplus value, $s > 0$, achieves the same minimum at *m*. But because surplus value, like a value-added tax, is charged only once on labor no matter how early is its date, the markup is

¹⁷ This section can be skipped, or read following the reading of Part Two.

the same percentage $(1+s)$, on all goods; hence the shift of the worker's budget-equation is inward in a parallel way. And of course there will be a unique s^* that produces the indicated broken line through *m* parallel to *ABZ*. The difference between the slopes of the two lines through *m* depicts the "contradiction" between Volume III's bourgeois prices and Volume I's Marxian values (only the latter having ratios that still agree with the ratios of the embodied labor requirements of the undiluted labor theory of value).

Figure 2 reveals even more explicitly the determination of the unique profit and surplus-value rates. Figure 2(a) presents the "factor-price frontier"¹⁸ relating the profit rate, *r*, to the real wage in terms of the first good, $W/P_1 = 1/A_{01}(r)$. Figure 2(b) relates *r* to the real wage in terms of the second good, $W/P_2 = 1/A_{02}(r)$. Figure 2(c) relates *r* to the (m_1) market basket of subsistence goods, $W/(P_1m_1 + P_2m_2) = 1/[A_{01}(r)m_1 + A_{02}(r)m_2]$. The exploitation equilibrium profit rate, r^* , is determined by the specified subsistence level *m*, as indicated at e^* .

The right-hand side of Figure 2 shows the same story, but this time told for "values." Now at the extreme right, in 2(d), the real wage of the first good is plotted against the rate of surplus value *s*, and is a rectangular hyperbola determined by the equation $w/p_1 = 1/A_{01}(0)(1+s)$, where values are indicated by the lower-case letters p_j . Similarly, 2(e) plots *s* against the real wage of the second good, expressed in the values regime. Finally, Figure 2(f) plots the real wage in terms of the market basket of subsistence wage goods, or mathematically $w/(p_1m_1 + p_2m_2) = 1/[A_{01}(0)m_1 + A_{02}(0)m_2](1+s)$. Notice that all the three right diagrams of the values regime are rectangular hyperbolae and so ratios of them all cancel out the $(1+s)$ = common factor and do reflect embodied labor coefficients.

¹⁸ For this concept, see Samuelson [32, 1962; 28, 1957; 27, 1970 and 31, 1961] or Sraffa [35, 1960].

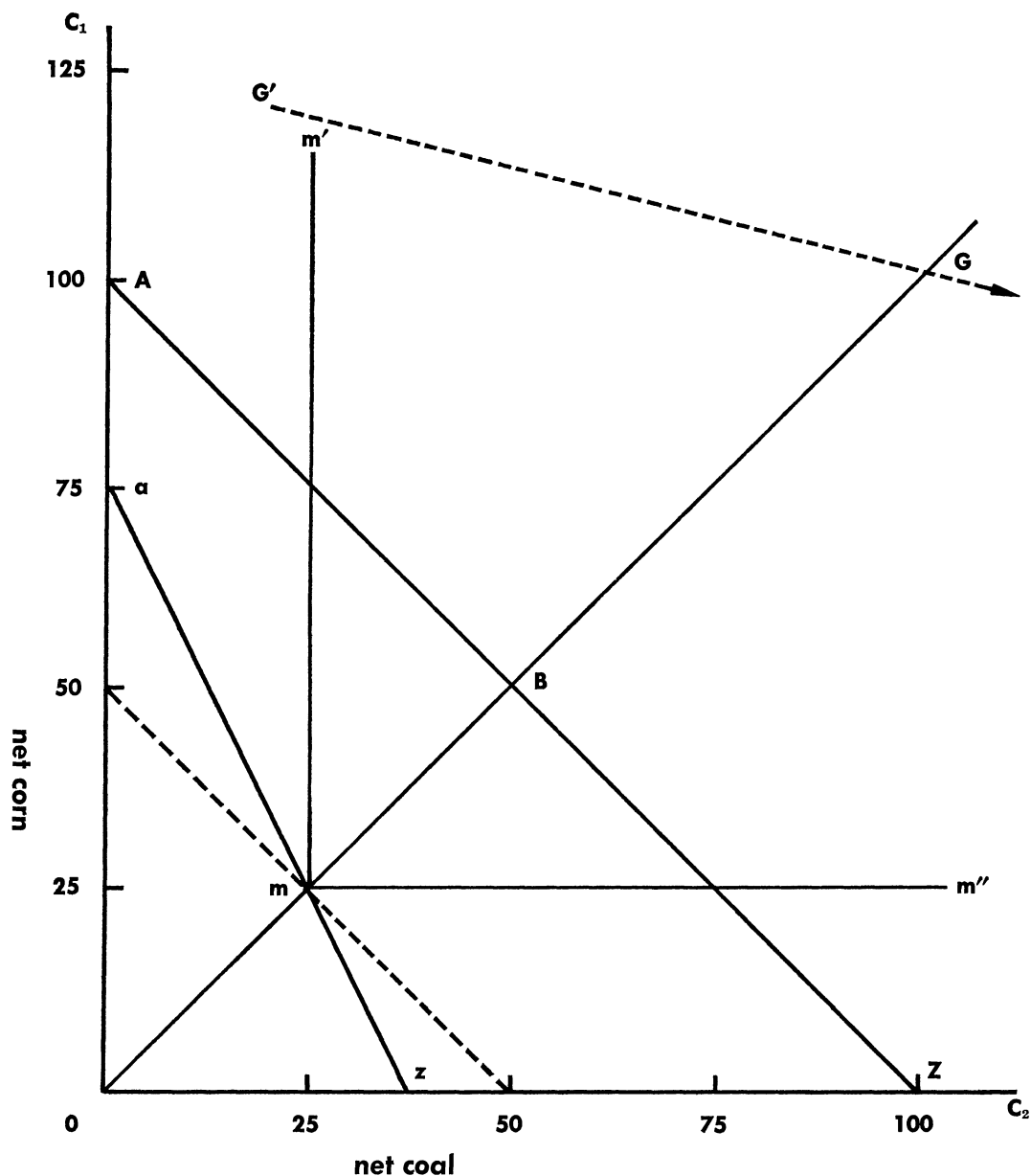


FIGURE 1

The production-possibility frontier of steady-state corn and coal, producible net by 100 labor, is given by ABZ , with slope reflecting total-embodied-labor hours, $A_{02}(0)/A_{01}(0)$. Subsistence specified as needed for reproduction of laborers is given by $m'm''$. A planned economy might go directly to m by command. Or, in a regime of Volume I's values raising the rate of surplus value, s , would shift the workers budget constraint inward in a fashion parallel to ABZ —until at $s^* = 100$ percent we reach the broken line through m parallel to ABZ . In a regime of Volume III's prices, raising the profit rate above zero shifts the budget constraint inward, in a steepened fashion relative to ABZ —until, at $r^* = 33 \frac{1}{3}$ percent, amz is reached, with price of coal risen relative to price of corn because of the latter's relatively greater ratio of direct labor and the implied greater $A_{02}(1/3)/A_{01}(1/3)$ ratio. Because of the singular assumption of equal-internal-compositions, the gross amounts of productions are shown at G , on the same ray as Om and OB .

A natural way to pair (r^*, s^*) values is shown by the horizontal real-wage line that is common to the two regimes. As that horizontal line is raised or lowered, we generate all the (r^*, s^*) pairings that correspond to the same real wage (in terms of specified m_j proportions of subsistence). Obviously, r^* and s^* rise and fall together. Obviously, $r^* < s^*$ if any early-date labor (constant capital) is involved in producing the subsistence goods. Obviously, changing the composition of the subsistence market-basket of goods will alter the pairing relationship between (r^*, s^*) .¹⁹

In the general case the Volume I relations on the right are simpler than those on the left. Being simple hyperbolae, they can be inverted and solved for s^* by a simple linear equation. The relations on the left involve Leontief-Sraffa compounding of profits and have to be written as ratios of a common n th degree polynomial to different n th degree polynomials for each good depending upon its time-profile of dated labor.²⁰

We have now completed Part One's preparatory review and background exposition of the regimes of prices and values. We are now in a position to review the troops, interpreting what Marx proposed and what each

¹⁹ The only exception to this is where labor intensities, a_{0j}/A_{0j} or organic compositions of capital, are the same for every good. In this singular case, all the factor-price frontiers are of the form $b_j(1-r/r_m)/(1+r)$ —becoming straight lines in the Sraffa [35, 1960, p. 30] fashion of a standard commodity, it being understood that the wage is expressed in end-of-period postfactum rather than usual beginning-of-period terms.

²⁰ *I.e.*, $a_0(1+r) [I-a(1+r)]^{-1}$ has elements which are the ratios of such polynomials in r . When there are two goods, the frontiers can change curvature; with n goods, there can be more inflection points; but always the frontier must decline monotonically. And if any necessary good requires, directly or indirectly, something of itself for its production, the frontier will intersect the horizontal axis at an r_{max} intercept, which will be the same for all such necessaries and which will equal the maximum rate of growth that the system is capable of even if labor were in unlimited supply. To sum up, solving for the exploitation profit rate r^* and the corresponding real wage(s) generally involves solving an n th degree polynomial for its smallest positive root, the algebraic counterpart of finding the proper intersection of a horizontal line with a declining frontier.

writer has added to the literature. In a sense these first remarks of mine have added a further contribution to the literature that can stand by itself as an elucidation of the exploitation-theory “transformation problem.” The results of this investigation can be recapitulated as follows.

1. If given the direct labor coefficients, a_0 or $[a_{0j}]$, and the input-output coefficients, a or $[a_{ij}]$, and the subsistence-wage parameters, m or $[m_i]$, direct planning and command can determine the allocation of final goods and real incomes between workers and non-workers (as in Figure 1).

2. Alternatively, one can postulate a Marxian rate of exploitation or surplus value, common to every industry, and easily solve for the s^* equilibrium rate and the implied “values”—which will be seen to involve ratios still equal to embodied labor requirements of the undiluted labor theory of value.

3. If one regards such Marxian markups as completely unrealistic in comparison with equal rates of profit on all the cost outlays of the industry at any and every stage of production, one can make a fresh start and deduce the exploitative rate of profit and competitive prices from the (a_0, a, m) data of the problem—making no use of surplus values or of Marxian values concepts. The line amz in Figure 1 shows the resulting equilibrium.

Looking at the last two paragraphs, one realizes that “transforming from values to prices” does literally involve “abandoning the values schemata of Volume I and embracing instead the prices schemata of Volume III and of bourgeois economics”—but of course with the understanding that the bourgeois tools are applied to the non-bourgeois hypothesis that workers end up working some of the hours of the day for their bare subsistence and the remaining hours of the day for the exploiting capitalists. By symmetry, the “inverse transformation from prices to values” involves “abandoning

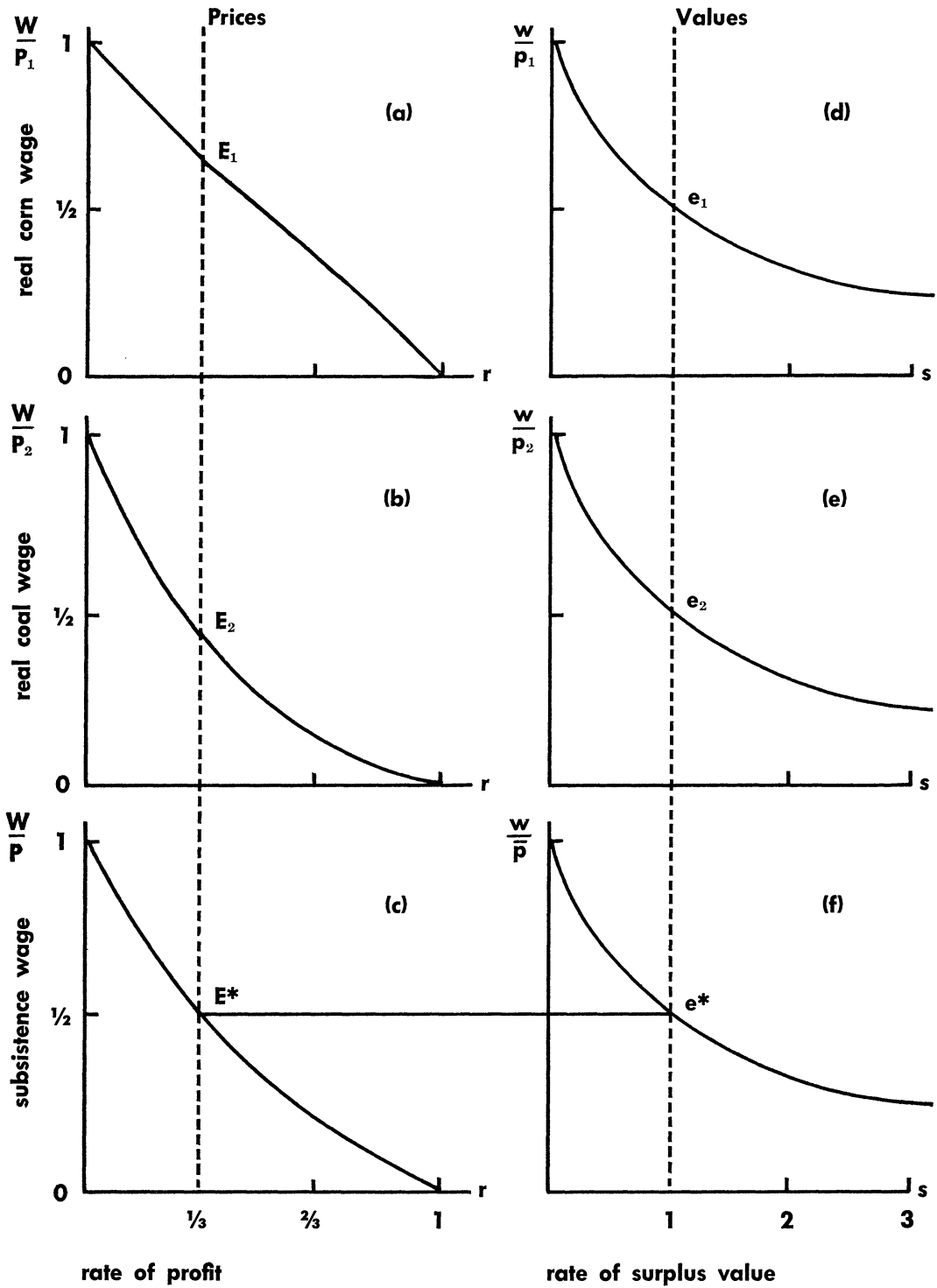


FIGURE 2

TABLE 1. MARX'S OWN TRANSFORMATION PROCEDURE

	Capitals or Cost Outlays (1)	Surplus Values (2)	Values (3) = (1) + (2)	Rate of Profit (4) = (2)/(1)	Prices (5) = (1)(1+.22)	Deviations of Prices from Values (6) = (5) - (3)
I	$80c_1 + 20v_1$	$20s_1$	120	20%	122	+2
II	$70c_2 + 30v_2$	$30s_2$	130	30%	122	-8
III	$60c_3 + 40v_3$	$40s_3$	140	40%	122	-18
IV	$85c_4 + 15v_4$	$15s_4$	115	15%	122	+7
V	$95c_5 + 5v_5$	$5s_5$	105	5%	122	+17
Average	100	22	122	22%	122	0

the price model in favor of working out the values directly from the (a_0, a, m) data.”

*Part Two. Marx's Transformation Process
And The Model of Exploitation*

*VI. The "Transformation" Procedure
of Volume III*

If actual market competition causes the rate of profit to be the same in all industries, it has been established that competitive "prices" will necessarily differ from the Marxian values (save in the singular case where all industries happen to have the same "organic composition of capital," i.e., the same ratio of direct wage costs to other costs). In Volume III of *Capital*, Marx faced up to the contradiction, indicating in a

well-known table how one "transforms" values into prices.²¹

Ever since 1894 there has been a good deal of commentary on this problem and its proposed solution. None of the writers, Marx included, were satisfied with the suggested procedure, which can be illustrated by the above adaptation of the tables from Volume III.²²

²¹ As I indicate in my bibliographical Part Three, the classical reference on this is Sweezy [36, 1942, Chapter VII], where the various writings are reviewed and a full discussion of the 1907 solution of Bortkiewicz [3] is explained. See also Sweezy [37, 1949], which provides an English editing of Böhm-Bawerk's, *Karl Marx and the close of his system* and Rudolph Hilferding's, *Böhm-Bawerk's criticisms of Marx*, and, as an Appendix, an English translation of Bortkiewicz [3]. See also another 1907 paper by Bortkiewicz [4], a treatment in some ways more fundamental; also note Winternitz [40, 1948], May [18, 1948], Joan Robinson [25, 1950], and Dobb [9, 1955]. Especially valuable as well is Meek [19, 1956], to the later printing of which I shall key my references. The first landmark, since 1907, in the analytical history of the subject is provided by Seton [33, 1957]. See also Morishima and Seton [21, 1961], Johansen [11, 1961], and my paper [26, Samuelson, 1970]. For the modern reader the well-known works of Leontief [13, 1941 and 14, 1966], Sraffa [35, 1960], and Dorfman-Samuelson-Solow [10, 1958] provide relevant techniques.

²² Table 1 is taken from the tables on pp. 183-85, Chapter IX, of the 1909 Kerr edition of *Capital*, III, except that I have corrected an obvious misprint of 150 for 105 in the first of those tables; I have also ignored Marx's complication in which all of constant capital is not used up in one period's production—so the reader can, if he wishes, subtract from the numbers in my

At the left are shown the downward-sloping factor-price frontiers, W/P_i , for each profit rate r . In (c), the condition that 100 labor be fed 25 of corn and of coal, out of the 50 of each producible net, determines $r^* = 1/3$ at E^* . Running over to e^* in (f), we similarly determine $s^* = 1$; running up vertically to e_2 and e_1 , we read off corresponding values, $w/p_i = 1/[A_{0i}(0)(1+s^*)]$. Either regime is computable without the other. If the a_0, a, m coefficients could take on more than one value, the outer envelope of such relations would be relevant, and the transformation from s^* to r^* , and vice versa, would be a more complicated problem than mere extraction of algebraic roots.

Marx here assumes five industries or departments. Direct wage payments (so-called variable capital) are given by 20 of v_1 , 30 of v_2 , . . . , etc; payments for intermediate goods (so-called constant capital) are given by 80 of c_1 , 70 of c_2 , At first he assumes a constant *rate of surplus value* (mark-up on wages) of 100 percent and derives his “values.” When he relates the total of the surplus, not to the total of wage payments alone (variable capital) but to the total of all cost outlays (constant capital plus variable capital), he finds divergent rates of profit being earned in the various industries with their different labor intensities; these average out to 22 percent.²³ So in column (5) are computed the competitive prices (what we would call the Walrasian equilibrium prices). Industries like V, which are not very labor intensive, are seen to have prices high relative to their values because of the fact that their prices must carry the heavy weight of profit earned on much capital. But, for all the industries of society, Marx has *by postulate* made prices average out to equality with values.

Without exception, critics and defenders alike, have recognized that Marx was not consistent in all this. For he mistakenly kept the same constant capitals, c_j , in his price calculation as well as in his value calculations. But what are the c s? They are the items that have been produced in earlier stages of production, and the same logic that causes values to be changed into prices requires that *their* values also be converted

into prices. Thus, it is argued, Marx went only part of the way and erred in retaining some elements of values calculation in arriving at his prices.

I must agree with this. Indeed, in some of his passages Marx indicates an explicit awareness of the inconsistency;²⁴ all the writers since 1907 have resolved the error by some variant of the procedure recommended by Bortkiewicz. Without dissenting from this consensus, my first effort here is to point out that there is a singular case in which Marx’s algorithm happens to be rigorously correct.

This singular case is worthy of explication in its own right as a *curiosum*. But much more important, by considering this case one can—in the words of the master himself—illuminate and strip bare some misconceptions concerning the sense in which one finds it necessary or useful to proceed from Volume I’s analysis of interindustry *values* in order to understand the nature of exploitation actually occurring in a competitive world of prices and as possibly giving a clue to the dynamic laws of motion of exploitative capitalism. I hope to demonstrate that anyone who believes in the relevance of a minimum-subsistence wage (I myself do not, either theoretically or empirically, but that is not relevant to the present effort) will understand his own theory better if he preserves from Volume I only the spirit of the insight that there is a discrepancy between what can be produced and what constitutes

column’s (3) and (5) the respective numbers [30, 19, 9, 45, 85] to get Marx’s more complicated second table. Unfortunately for the reader in a hurry, the literature has mostly concentrated on the more complicated case, which merely slows down but does not alter the analysis. (Paging in the Kerr edition runs about 30 longer than in some other editions; citation of chapters, however, should enable one to find any quoted passages.)

²³ If Marx had not chosen all $c_j + v_j$ equal, the average profit would involve using such column (1) items as weights instead of being computed, as here, by a simple unweighted average of the separate industry profit rates.

²⁴ In *Capital*, III, Chapter IX, p. 194, Marx notes: “Now the price of production of a certain commodity is its cost-price for the buyer, and this price must pass into other commodities and become an element of their prices.” At least one writer [19, Meek, 1956, 148ff] seems to believe that if Marx’s table can ignore “mutual interdependence,” no error is involved. Not so: even if the c of Department I comes from that department alone, or from an “earlier stage” of it, profit calculations will alter the c that had been appropriate for surplus-values calculations. So the same problem arises here that Meek and all the writers recognize in connection with models of simple or extended reproduction where mutual interdependencies are obvious.

the minimum wage, and he will do better to jettison as unnecessary and obfuscating to his *own* theory the letter of Volume I's analysis of inter-industry values.

VII. *The Singular Case of Equal Internal Compositions*

The singular case in which Marx's procedure becomes exact can now be described. It is not the well-known case where all labor intensities are equal (the case of equal organic composition of capital) for in that case the problem becomes transparent and trivial, there being no longer any "contradiction" between values and prices. If the world were like that, Marx would have had no reason to try to improve upon the bourgeois economists' analysis of prices, since relative prices and values are then identical.

Instead we may now consider what might be called the case of "equal *internal* compositions of (constant) capitals." In this case *every one of the departments happens to use the various raw materials and machine services in the same proportions that society produces them in toto*. Thus, if the five departments represent different goods (say corn, coal, . . .) then each department must have the same ratio of corn to coal in its constant capital as every other department. This technological requirement is, of course, not particularly realistic (although Leontief's input-output matrices do exhibit some curious similarities of columns) and its disparities from realism help to elucidate the objections to Marx's procedures more cogently than do many of the sometimes sterile commentaries on him. A second postulate goes along with the above technological condition; again it is not a very realistic one: we must also assume that the *minimum-subsistence budget is a market basket of goods that comes in those same relative proportions as the goods are used as inputs in production*. (This is because a subsistence-wage theory is somehow assuming that the labor supply itself is, as it were, *produced* by a

further department not all that different from other departments. Marx, it will be recalled, did not like Malthusian population arguments; but he was a classical economist who believed in a cost of production for labor power itself, even if he enunciated less clearly than von Neumann has in our day the linkages of this mechanism.) In the end it will be observed that the capitalists, who get what is left over after the specified requirements of industries and wage-subsistence have been met, will also receive goods in these same proportions. However, this is a theorem of our analysis and not a separate postulate, being already implied by our other postulates. The capitalists are free to devote these goods to luxury consumption (as in a Marxian model of "simple reproduction") or in part or whole to accumulating the increments of physical constant capitals needed to provide for balanced exponential growth (as in Marxian models of "extended reproduction," or so-called golden ages). The fact that the capitalists use goods in the same proportions as the workers is evidently at variance with those reproduction models of Marx carried over from Volume II, in which separate wage-good, luxury-good, and producer's-good departments are postulated. And this alerts us to the fact that almost all writers, though they begin with the five departments that Marx himself began with, choose somewhat gratuitously to rush on to apply the procedure of Table I to Marx's models of simple reproduction²⁵—a legitimate procedure but one

²⁵ In the paper [3, Bortkiewicz, 1907] upon which Sweezy and later writers primarily rely, this is done. Bortkiewicz even presumes to alter some of Marx's numbers in his other 1907 paper in order to make the identification. In this he seems merely to be following in the tradition of Tugan-Baranowsky and other contemporary writers. But in his other paper, he clearly shows that he does understand how to handle the general case, indeed indicating a better understanding of the nuances of the problem than those who have essentially adopted his solution; I cannot help feel that Bortkiewicz has not been given full justice by many subsequent writers, who may have been put off by his tart defense of Ricardo against Marx's criticisms, and

that Marx himself was, as far as I can remember, too wary to attempt.

VIII. Marx Vindicated

Now let us return to Table 1 and note how our singular case of constant-internal-compositions-of-goods does validate Marx's simplistic procedure. If the 80 of c in Department I is in fact made up of a weighted combination of column (5) prices that average out to the same as column (3) values, and only if this is the case, we can be sure that 80 remains the right magnitude for both the price and value calculations. We prove this by trial substitution of the asserted prices and verification that the process is indeed self justifying. We also verify something that Marx and most pre-Seton writers other than Bortkiewicz failed to emphasize sufficiently—that the real wage has indeed been kept to the same exact level in both the mode of prices and of values. Thus, without my postulate, Marx's procedure is open to the further fatal objection, namely that the market basket of subsistence will not cost the same in the two regimes relative to an hour's labor. (When Marx tells us that the workers work half the day for themselves and half the day for the exploiting capitalists, he does not indicate in his various tables

by his tireless analysis of every detail. As is indicated in various parts of the present paper, I cannot agree with the mathematician May that Bortkiewicz's mathematics is overly elaborate, or dependent on overly-strong sufficiency conditions, or that he has introduced pseudo-mystifications, or that he has adopted normalization rules for the absolute price level that are clearly inconsistent with Marx. As May has suggested, to the pure mathematician the algebra of eigenvalues is trivial, as is most of physics and engineering. Still, such eminent men as Perron, Frobenius, Markov, Fréchet, Minkowski, and Besicovitch have devoted themselves to his area, and it is not until the writings of Seton and Morishima that one finds in the modern literature a full understanding of Bortkiewicz. As Sweezy, the one Marxian writer who does express appreciation for the Bortkiewicz contributions, has pointed out, Bortkiewicz did glean from the Marxian analysis the essential point—that setting the real wage at a specified subsistence does provide a determinate profit level at which labor's full product gets "discounted."

what fraction of the goods in the different departments make up their iron ration. If we ourselves pick such proportions at random, there is no reason why the properly *weighted* average of the new prices should at all come to the same level as for values even though that is true for unweighted averages. This would mean that Marx's 22 percent profit rate is the wrong rate for him to use. In the modern era, Leontief and Sraffa, Dorfman-Samuelson-Solow, Seton, and Morishima know how to calculate the polynomials that give the proper departure from 22 percent; so too does Bortkiewicz as a result of his perusal of the work of the young Russian Dmitriev and his understanding of Walras. But the other writers, by rushing from the five department case to models where one industry produces all the wage goods, hardly show an awareness that there is a problem here. Instead they waste much time on the unessential question of what *absolute* level of prices should one introduce into a table whose only importance consists in its well-determined *relative* proportions!)

But to repeat and summarize, on the basis of my singular case of equal-internal-compositions, Marx has been preserved from all pitfalls.

IX. The "Inverse Transformation" Problem

Table 1 now well illustrates that one could as easily start with prices and perform what Morishima-Seton call the "inverse transformation" from prices to values. And again, the simple procedure indicated by Marx's school arithmetic will be rigorously correct in my singular case (and only then).

Let us analyze the steps in going from columns (1) and (5) back to columns (3) and (2). The *total* discrepancy between the final prices of (5) and the cost outlays of (1) is given by $22 + 22 + 22 + 22 + 22$; these numbers are constant only because of Marx's penchant for simple examples that

involve hundreds, but in any case one can compute ΣS_j (reckoned in price units of course). This can be equated to Σs_j reckoned in value units.²⁶ Since Marx's case involves no change in the totals of values and prices, the rate of surplus value, s , is evenly computed and columns (2) and (3) can be filled in. But now, just as Marx turned Hegel on his head, we turn Marx upside down, and can say in a dozen repetitive ways that this total of *profit* is not allocated by the *value* system according to where it was "really produced" but rather "falls to the share of each aliquot part of the total social [variable] capital out of the total social profit [!]." Each time that I have inverted prices and values, rates of profit and surplus value, variable and total capital, I have introduced the bracketed interjections. To parody Marx's own words further:

Surplus value [!] is therefore that disguise of profit [!] which must be removed before the real nature of profit [!] can be discovered. In profit [!], the relation between capital and labour is laid bare [III, Chapter II, p. 62]. . . . [without Volume III's [!] analysis of profit [!], political economy would be deprived] "of every rational basis". [III, Chapter VIII, pp. 176-77] . . . [and the average rate of surplus value [!] would be] an average of nothing (*Theories of surplus value*, p. 231, as cited by Meek [19, 1956]).

A little of this parodying goes a long way. A reader who refers to Meek's careful elucidation of what Marx seems to have intended will find many more passages open to the same mishandling. But to Marx these

²⁶ The average rate of surplus value is given by $s = \Sigma S_j / \Sigma V_j = 110/110$, or 100 percent. Now we calculate column (2) by $s_j = (1+s)V_j = (1+s)v_j$, with $\Sigma s_j = \Sigma S_j$. And finally we get the values of (3) from $c_j + v_j + s_j = c_j + v_j + sv_j$. Note two things; this simplified inverse operation is open to all the original objections against Marx's neglect of the changes in the constant capitals in going from one system to the other, and can be defended only if my singular assumption is made. Also, note that $s = \Sigma S_j / \Sigma V_j$ is not the *simple* average of the (S_j/V_j) 's, namely $\Sigma (S_j/V_j)/5$, but is rather the *weighted* average $\Sigma V_j(S_j/V_j) / \Sigma V_j$, where the weights are the variable capitals (expressible in either units when my singular assumption is posited).

were no joking matters. For we are close to the heartland of what he regarded as his theoretical innovation.²⁷ In a proud passage, he claims that

the actual state of things is revealed for the first time; that political economy up to the present time . . . made either forced abstractions of the distinctions between surplus-value and profit . . . or gave up the determination of value and with it all safeguards of scientific procedure, in order to cling to the obvious phenomenon of these differences—this confusion of the theoretical economists demonstrates most strikingly the utter incapacity of the capitalist, when blinded by competition, to penetrate through the outer disguise into the internal essence and the inner form of the capitalist process of production [III, Chapter IX, p. 199].

X. A Misunderstanding

One common misunderstanding of the inverse transformation problem needs to be cleared up. It is common to a critic of Marx like Böhm-Bawerk and a sympathizer like Professor Joan Robinson. In effect, both argue that only profits and prices have a reality and that Marx in beginning with values and rates of surplus value has *already* performed the inverse transformation; thus the direct transformation merely brings him back to his starting place. Mrs. Robinson put it this way:

Sweezy . . . evidently fails to realize that the transformation problem and its resolution is just a toy and that the whole argument is condemned to circularity from birth, because the *values* which have to be "transformed into prices" are arrived at in the first instance by transforming prices into values. [25, Robinson, 1950, p. 362].

This is simply incorrect. Mrs. Robinson, not knowing in 1950 the Sraffa apparatus,

²⁷ Friedrich Engels concurred with this view that the transformation problem was at the heart of Karl Marx's basic contribution to economics. In the 1885 preface to *Capital*, Volume II, Engels had already thrown out the challenge to writers to anticipate the reconciling of prices and values that was to be forthcoming in the third volume. And in the 1894 preface to that latter work, he comments scathingly on proposals to resolve the riddle, comparing them unfavorably with the revelations in Volume III itself.

or the equivalent Leontief apparatus, has failed to realize that one can go from an undiluted labor theory of value, in which the direct plus indirect labor-hour requirements of each good and the subsistence can be reckoned up in physical terms to Marx's tableau of values. And we do so without having to solve any high degree polynomials (for five departments, generally a fifth degree polynomial) as we should have to do to arrive at the tableau involving profits and prices.²⁸ Having said this, let me hasten to say that I agree with her general point of view, most particularly with the words that follow on the same page, namely that Marxian exploitation is to be conceived "in terms of the division of working time into the part necessary to procure the subsistence of the workers, and the rest, which produces surplus. *This has no meaning as applied to separate industries.*" (Italics added.)

Böhm-Bawerk, in the cited piece on the conclusion of Marx's system, and also earlier in his writings during the 1880s on the exploitation theory of interest [1, Böhm-Bawerk, 1959], has made many valid points against Marx, for which sincere Marxian scholars should be grateful rather than otherwise. But he does not do the theory of exploitation full justice on one important logical point. Repeatedly he points to the admissions by Marx and his sympathizers that equality of profit rates and the implied prices are more realistic than the equality of rates of surplus values and the implied values. He rightly regards this as a rejection of the labor theory of value in the sense that the realistic exchange ratios are not to be inferred from ratios of necessary

²⁸ I exaggerate—a little. In making up one's own arithmetic examples, one can cook the coefficients of the polynomials to produce a prescribed simple root. And one can make singular simplifications, like mine here; or can assume decomposability of the system into somewhat independent parts, etc. The point remains though: once you know total labor embodiments, values are easy for the non-mathematical to calculate, and prices are not—which is a theorem about certain matrix equations.

labor time. But it is a non-sequitur for him to think that all this has negated the labor theory of value in its undiluted, technocratic sense, namely as an algebraic procedure that tells you what can be produced with a given total of labor in the steady state when all capital goods have become adjusted.²⁹ As will be seen in the following, a theory of the exploitation wage can be based solely upon the analysis of profits and prices (with all the Leontief-Sraffa algebraic complications that are implied by these real-world relations). And some day Marxians will probably wish to formulate it in those terms. But this does not deny that this same level of subsistence wages *could* be arrived at in a regime which chose to organize its exchange relations on the basis of Marx's Volume I hypotheses. (Recall Figure 1 of Part One in in this regard.) One might apply Marx's theory of the materialist determination of history to arrive at the hypothesis that it was Marx's incapacity in algebra and the absence of a computer that caused him to formulate his exploitation theory in Volume I terms which are unrealistic but which happen to be simpler to handle algebraically than Volume III's Walrasian relations.

XI. Redundancy of Industry Surplus Values

By now the crucial issue is no longer whether Volume III's prices are more realis-

²⁹ Böhm-Bawerk correctly points out that primary factors other than labor can affect market scarcities as these work themselves out in terms of supply and demand. And with generosity we can construe him to be pointing out that, when you do not confine yourselves to steady states, the supply conditions of stocks of heterogeneous capital goods also affect scarcities as these work themselves out in supply and demand markets, so as to produce exchange relations that are at variance with simple labor-theory-of-value notions. But these cogent criticisms should not be thought to vitiate the logic of an exploitation theory based upon the discrepancy between what labor can produce in the steady state and a minimum-subsistence level at which labor power can be reproduced at will. *I.e.*, Marx's version of the von Neumann model is not illogical, despite Böhm's charges that his opponents are renegeing on their own rejection of the labor theory of value

tic under competition than Volume I's values: critics and Marxians are agreed that these prices certainly are. The issue has been narrowed down to whether, as Marx and his modern defenders have claimed, the profit rate upon which Volume III's Walrasian equilibrium depends is itself crucially determined by Volume I's analysis of surplus values, $s_j = v_j s$, or crucially dependent on the *totals* of these magnitudes (in the sense that the profit rate r can only be calculated *after* these have been summed up and averaged out).

First, this section demonstrates that Marx and Engels—and, in modern times, Dobb and Meek—are simply wrong in their identification of what aspect of the labor theory of value is intrinsically involved in working out a price-profit configuration that corresponds to the minimum-wage theory of exploitation. The most clear-cut proof is mathematical.³⁰

Since there is no reason to expect a scholar interested in the process of exploitation and the laws of motion of developing capitalism to be also a virtuoso in matrix algebra, the second demonstration is by means of a simplified two department numerical example. Third, graphical analysis will clinch the point that profit-price equilibrium is determinable *solely* from the production coefficient specifying the required labor embodiments of the industries and from the minimum subsistence wage-good requirements. At no stage of the argument is there

³⁰ As has been already indicated in Part One, from knowledge of technology alone we know the a_0 and a coefficients. After these have been supplemented by specified subsistence requirements, m , Volume III's prices stand on their own feet and are defined by $P/W = a_0(1+r)[I-a(1+r)]^{-1} = A_0(r)$, $mA_0(r) = 1$. These relations require no prior consideration of the Volume I value relations, $p/w = a_0(1+s)[I-a]^{-1} = A_0(0)$, $mA_0(0)(1+s) = 1$, and no transformation procedure in either direction. Cf. Samuelson [26, 1970], equations (3) and (2), and the demonstration in (5) that Bortkiewicz goes from values to prices by cancelling out values! In Bortkiewicz [3, 1907] equations (20) and (28) are equivalent to the above price relations, and equations (9) and (11) equivalent to the above value relations.

need for, or benefit from, utilizing the magnitudes $c_j + v_j + v_j s$ of Volume I.

Numerical Example. Let society's 100 units of labor be allocated so that 80 work in Department I (say, corn) and 20 work in Department II (say, coal). In I, 100 units of corn are produced by 80 units of labor and 10 of corn and 10 of coal. In II, 100 units of coal are produced by 20 of labor and 40 of corn and 40 of coal. Let the minimum-subsistence wage require that each labor unit consumes a market basket of 1/4 final units of corn and 1/4 final units of coal, so that 25 units out of each 100 of gross outputs go to workers. Since another 50 of each is seen to go for intermediate products, that leaves 25 of each for non-workers' final consumption or saving-investment.

Note that my singular assumption of equal internal compositions is realized here. Note, too, that this physical tableau can be specified completely independently of either Volume I values or Volume III prices and of profit rates or surplus-value rates. It could have been achieved by a computer-guided command economy using planning and direct rationing.

Now consider a Volume I version of this same situation. The crucial ratio of what labor produces in comparison with what it needs for subsistence has already been determined by the physical tableau *prior* to any calculation of industry surplus values or totals of these magnitudes. We can translate these 25/50 relations into Marx's rate of surplus value by setting $s = 100$ percent. As shown in Table II's second and third columns, we can calculate the exchange ratios that prevail on the bizarre empirical hypothesis that the system is organized so that it is not the rate of profit which is equalized between industries, but rather their rates of surplus value s_j/v_j .

Alternatively, could we have *first* written down the Volume III price-profit relations shown in Columns (4) and (5)? For Marx or Meek to be making a valid defense of the

TABLE 2. EQUAL-INTERNAL-ORGANIC CASE OF SIMPLE REPRODUCTION

	Capitals (1)	Surplus Values (2)	Values (3) = (1) + (2)	Profits (4) = 1/3(1)	Prices (5) = (1) + (4)	Deviations (6) = (5) - (3)
I	$(20+20)c+80v$	$80v$	200	40	160	-40
II	$(80+80)c+20v$	$20v$	200	60	240	+40
$r = \sum s_j / \sum (c_j + v_j) = 100/300 = 1/3$						0

Explanation: prices and values of each of 100 units of corn, and of coal, are respectively $(P_1, P_2) = (1.6, 2.4)$ and $(p_1, p_2) = (2.0, 2.0)$. The valuations are applied to physical amounts of corn and coal used as intermediate goods; wage = 1 throughout. When prices are applied in (1), we get $(16+24)c+80v$ and $(64+96)c+20v$ instead of the numbers shown there, but with the same *totals*.

important role of Volume I's analysis of surplus values, the answer to this question would have to be in the negative. But, clearly, there is one and only one profit rate, namely $33 \frac{1}{3}$ percent, which could allocate to labor the specified minimum real wages. Even one who knows no algebra can satisfy himself of this by trying all possible profit rates and working out the resulting Columns (4) and (5). For *any* $r > 33 \frac{1}{3}$, the prices charged each worker would be too high to permit him to buy the specified $1/4$ units of corn and $1/4$ units of coal with the wages earned by a unit of his labor time. For *any* $r < 33 \frac{1}{3}$, his real wages would prove to be too high. So Columns (4) and (5) are definitely computable *prior* to the calculation of any item in Columns (2) and (3)—as shown. Q.E.D.

Graphical Demonstration. The reader can now refer back to Figure 1 to see the same point clearly. The production-possibility frontier of final goods, as given by *ABZ*, is based solely on the embodied labor requirements (as determined by the numerical $[a_0, a]$ coefficients: 80/100, 10/100, 10/100; 20/100, 40/100, 40/100; and nothing else). The minimum-subsistence wage requirement, it will be recalled, is shown by the

L-shaped indifference contour, $m'mm''$ (as determined by the 100/4, 100/4 coefficients, m , and nothing else).³¹ So far neither prices and profit rates nor values and surplus-values are involved.

Now, to drive home the crucial point, this time consider Volume III's case first. If the profit rate had been zero—as in Adam Smith's "early and rude state" where the undiluted labor theory of value gives all to workers and leads to exchange values not merely *proportional* to embodied labors but *equal* to those embodied labors and hence equal to bourgeois prices—the budget line for the workers would have been the *ABZ* frontier itself. Then, as profit becomes positive and r is increased, their budget line naturally shifts inward by virtue of prices being marked up relative to the wage rate. However, as Ricardo conceded, the inward shift will not be parallel with all *P*s rising in the same proportion; instead, corn being

³¹ The gross amount of corn and coal produced, 100 units of each, is shown back in Figure 1 at *G*. In my singular case of equal internal compositions, m and G happen to lie on the same *OmG* ray (as, hence, must *B* also). But if non-workers were to shift their tastes away from *B*, as for example toward more corn, the point *G* would shift upward toward the point *G'*. Similarly, a shift in tastes toward coal would shift *G* oppositely, in either case destroying the simplifications of equal internal compositions.

more labor intensive will have its price fall relative to that of coal with its larger total profit markup. So each positive r gives a line steeper than ABZ , until at $r = .33 \frac{1}{3}$ we find the unique amz line that goes precisely through the prescribed m point—and which is seen for our numbers to have an absolute slope of $P_2/P_1 = 240/160 = 3/2$.

For one who believes in an exploitation theory of wages (and interest), the task is done. There is no need for, or interest in, Volume I's $c_j + v_j + sv_j$ analysis. And thus there is no need for the transformation problem, either direct or inverse. I think that Marx, were he alive today, would agree with this. In any case, Marxians in the future can be expected to agree with these prosaic and uncontroversial facts of arithmetic and logic.

For completeness, and possibly out of an antiquarian interest, we can also depict in Figure 1 the redundant Volume I calculations that lead to the broken line through m parallel to ABZ . But Mrs. Robinson's 1950 assertion that this must be done by means of the inverse transformation problem, in which one must obtain that parallel line by way of first calculating the amz line of prices and profit, is now seen to be incorrect. Instead—and here the Volume I algebra is admittedly simpler than that of Volume III, in the same way that the incidence of a value-added tax is simpler to calculate than that of a turnover tax—we raise gradually the rate of surplus value, s , above zero. Again the worker's budget line, of what they could buy in the market place at the quoted Marxian values, is shifted inward. But this time the shift is truly a simple parallel one! (Proof: an ever-larger fraction of the workers' hours available for their own consumption can be thought of as being taken away from them and being made subject to the consumption whims of the capitalist. And 99 hours of labors' time can produce for labor only 99/100 of *any and all* of the wage goods previously producible, etc.). Clearly,

only one parallel line will pass through m ; such a shift, half way to the origin in our example, must correspond to a rate of surplus value of exactly 100 percent.³²



The truth has now been laid bare. Stripped of logical complication and confusion, anybody's method of solving the famous transformation problem is seen to involve returning from the unnecessary detour taken in Volume I's analysis of values. As I have cited in my mathematical paper,³³ such a "transformation" is precisely like that in which an eraser is used to rub out an earlier entry, after which we make a new start to end up with the properly calculated entry.

XII. Conclusion On What Part of Marx Is Vital

Those who believe there is merit and importance in the notion of wages as determined by a-cost-of-production-of-labor-power will not infer that my dissection of Marx has robbed him of any credit for his essential insight. Marx needs to be protected from his defenders and occasionally from himself. (After all, did he not once say, "I am not a Marxist," a proposition that does not have to be applied only to the field of ideology and tactics).

³² Graphically one can read s and r off the diagram. Write the length of any line segment, such as mB , as $|mB|$. Then in Figure 1, $s = |mB| / |Om|$; and $r = |mB| / |Om + OG|$. In the general case of not-necessarily-equal internal compositions, where m , B , and G need not line up on the same ray, we measure s by passing lines parallel to ABZ and calculate the equivalent indicated distances between them. To calculate r , we use lines parallel to amz .

³³ The final words of Samuelson [26, 1970] read: "The present elucidation should not rob Marx of esteem in the eyes of those who believe that a subsistence wage provides valuable insights into the dynamic laws of motion of capitalism."

A conciliatory formulation that preserves honor all around would say:

Although Capital's total findings need not have been developed in dependence upon Volume I's digression into surplus values, its essential insight does depend crucially on comparison of the subsistence goods needed to produce and reproduce labor with what the undiluted labor theory of value calculates to be the amount of goods producible for all classes in view of the embodied labor requirements of the goods.³⁴ The tools of bourgeois analysis could have been used to discover and expound this notion of exploitation if only those econo-

³⁴ The following quotations from various Marxian authors will show that the present conciliatory formulation is consistent with their thinking. I hope that the present paper has been persuasive that going beyond this formulation leads to unwarranted assertions. In his 1865 *Wages, price and profit*, Section VIII [17, Marx and Engels, 1968] Marx clearly noted: "Now suppose that the average amount of the daily necessities of a labouring man require *six hours of average labour* for their production. . . . But our man is a wages labourer. He must, therefore sell his labour power to a capitalist [who will], therefore, make him work say, daily *twelve hours*. . . he will, therefore, have to work *six other hours*, which I shall call hours of *surplus labour*, which surplus labour will realize itself in a *surplus value* . . . [whose rate] will, therefore, depend on the *ratio in which the working day is prolonged*. . . ." Engels, in his 1891 introduction to Marx's 1849 *Wage, labour and capital* [17, Marx and Engels, 1968] similarly sums up the essence of Marxism: "With the present state of production, human labour power not only produces in one day a greater value than it itself possesses and costs; with every new . . . invention, this surplus of its daily product over its daily cost increases. . . ." Likewise, from Lenin in his 1913 "Three Sources and Three Component Parts of Marxism," Section II [17]: "The worker spends one part of the day covering the cost of maintaining himself and his family (wages), while the other part of the day he works without remuneration, creating for the capitalist *surplus-value*, the source of profit, the source of wealth of the capitalist class." See also the 1949 statement by Dobb [9, p. 111, n. 1] where he writes that the classical theory avers that ". . . the amount of profit (and given the stock of capital, the rate of profit) depends upon the ratio of the amount of labour time required on the average to produce a worker's subsistence . . . and the labour time worked by the worker." Having said this, Dobb would have done better to have omitted the earlier passage which asserts that ". . . exchange ratios were in the last analysis *determined* by the relative quantities of embodied labour in various commodities," a proposition clearly

mists had been motivated to use the tools for this purpose.

Leaving the realm of pure science, we may perhaps add that the path taken in Volume I, even if seen to be unnecessary in the present age, did have the advantage of being easier to expound logically. It also lent itself to a more emotive language that must have been influential in converting readers to the Marxian vision of the world. Even today, the most puristic scholar and teacher can fall back upon the Volume I terminologies with the best of conscience provided only that he prefaces his exposition with the observation that the case of equal organic compositions of capital or of labor intensities, although not particularly realistic, does provide a clear searchlight on the nature and dynamic development of a model of labor exploitation. Those who regard a subsistence wage model as a grotesque interpretation of history can also avail themselves of these harmless classroom simplifications in their critiques of the doctrine.

Part Three. Critical Review of the Literature

XIII. General Guide

A reader cannot do better than to begin with P. M. Sweezy [36, 1942], whose Chapter

denied by Figure 1's *amz*. Meek himself, in the course of an excellent 1961 review of Sraffa, *Production of commodities by means of commodities* [20, pp. 161–78], attempts to defend Marx's analysis in terms of Sraffa's "standard industry." But he does not explicitly note that Sraffa nowhere makes use of any of the Volume I $s_j = v_j s$ equivalences. Meek's own formulation of the transformation problem, as the careful reader of his long footnote 12 will discern, couched though it may be in terms of $\sum(v_j + s_j) / \sum v_j$ ratios, also has absolutely no need for these Volume I equivalences; in terms of my Figure 1, it is easy to show that the ratios that Meek does rely on are dependent only on the technical properties of *ABZ* and *m'mm'* and are determinable *prior* to computation of either Marxian values or competitive prices. This remark illuminates the fact that even if one swallowed with Marx and Engels the doubtful anthropology of an earlier golden age in which an undiluted labor theory of value applied, the relevant features of that regime would be described by bourgeois prices with no need for innovations of the Volume I type.

VII gives a lucid description of the problem, of Marx's proposed solution, and of Bortkiewicz's rectification of that procedure as applied to the case of "simple reproduction." He could follow this by reading R. Meek [19, 1956] who argues the strongest possible defense of the view that Marx's Volume I concepts were not redundant and obfuscating and who cites many of the divergent writings. Finally, the reader equipped with mathematics will wish to consult the important 1957 paper by F. Seton [33], the 1961 papers by Morishima and Seton [21], that by L. Johansen [11, 1961], and, possibly, my cited 1970 paper [26, Samuelson].

On Marx's procedure itself, Part Two of this paper provides a full discussion and list of citations. From the first appearance (1867) of the first volume of *Capital*, Marx's general viewpoint and concepts of surplus value came under attack by neoclassical economists. (Mill lived for a few years after 1867 and although he had followed closely the events of the 1848 Revolution, as far as I can recall he never made a single reference to Karl Marx.) Like Böhm-Bawerk [39, Wicksteed, 1933], Pareto [23, 1966] paid Marx the compliment of "refuting" him. Although some German socialists in the 1920s rejected Marxism as an alien "English" doctrine, Shaw was probably at least half right in claiming that he was the only man in England in the 1880s who had read and understood *Capital*. Hence, the magisterial Alfred Marshall could afford for the most part to ignore Marx. Knut Wicksell on the continent was more exposed to the Marxist ideology, but with the stubborn independence that was so characteristic of him, Wicksell espoused the cause of the working poor by advocating general reform doctrines that had much in common with the Roosevelt New Deal of thirty years later.

Some twenty years ago at a conference at American University, I touched a filial nerve in John Maurice Clark when I cast some

doubts about his father's belief that he, John Bates Clark, had irrefutably proved in the last decade of the last century the ethical justness of the marginal productivity mode of distribution. In his reply [41, D. M. Wright, 1951, p. 329, n. 14] J. M. Clark said that his father had been deeply conscious of the challenge offered by Marx's notions of exploitation ("under whose theory any share capital gets is outright robbery") and felt under a necessity to defend the competitive system from those charges, which if true would have admittedly constituted a grave indictment.

Marx had to be refuted by orthodox economists—if only because he was there! But to recognize this fact about ideology is not, in my view, to accept the notion that the merits of Marx's hypotheses and criticisms of them are incapable of being discussed rationally and objectively. Despite the class struggle, two and two remains four and not five; and an approximate answer to the question of whether real wages rise or stagnate over a century should be capable of being given to the satisfaction of a jury recruited in New York, Moscow, Delhi, Prague, and Peking. In any case, analysis can isolate the irreducible bones of contention.

On the narrower issues involved in the transformation problem itself, my earlier remarks have already suggested that I do not think that Böhm-Bawerk's critiques, either of 1898 [2], or in the final editions of his *Capital and interest* [1], added much; and they may, from the deepest view of the subject, have even subtracted a little. And I must agree with Joan Robinson in her evaluation of the cogency of Rudolf Hilferding's rebuttals to Böhm-Bawerk on this matter: "... though Hilferding scores one or two telling points against Böhm-Bawerk's own theory, he throws no light whatever . . . on the meaning of the theory that *value* determines price" [25, Robinson, 1950, p. 361].

XIV. The Bortkiewicz Contribution

As Sweezy makes clear, it was Ladislaus von Bortkiewicz who gave the transformation problem its definitive formulation. J. Winternitz errs in thinking that Bortkiewicz was an Austrian. A follower of Walras and an admirer of Ricardo, he was a rather cantankerous professor at Berlin who monitored—or better, policed—all the German literature of his day even to the point of recalculating regression coefficients to reveal their numerical inaccuracies. He was a gifted mathematical statistician who did important work in demography (and also worked in connection with the Poisson³⁵ analysis of rare events). He was a teacher of the young Leontief. His concern with Böhm's interest theory and his knowledge of the works of the Russians [38, Tugan-Baranowsky, 1905 and 8, Dmitriev, 1968] led him to write no less than three articles on Marx in 1907; as Sweezy notes, even at the late date of 1923, eight years before his death, Bortkiewicz was still writing on this subject.

Of his two papers published in July 1907, the better known one is that in the *Jahrbücher* [3, Bortkiewicz], which Sweezy has popularized and edited. Actually, however, the other and longer paper in the *Archiv* [4, Bortkiewicz], is the deeper of the two; it shows itself capable, in principle, of handling the general case, and not simply the special model of simple reproduction. When Sweezy states the belief that it does not attempt to solve the transformation problem as Marx himself presented it, he is perhaps being misled by the fact that no tableau of Marx is worked out in numerical terms. As indicated in my footnote 33, although Bortkiewicz never quite employs the Leontief *a* matrix, his equations (9) and (11) are equiv-

³⁵ To him is due the well-known example of the number of Prussian soldiers killed each year by the kick of a mule, and less happily, the designation of the Poisson distribution of rare events as the "law of *small* numbers" (which is not in legitimate contrast to the "law of *large* numbers").

alent to my "values" formulation and his equations (20) and (28) are equivalent to my "prices."³⁶

In accordance with the usual practice of writers, let me concentrate on the treatment involving simple reproduction. In this paper Bortkiewicz follows the example of Tugan-Baranowsky [38, 1905] and contemplates a Marxian model with three departments. Department I produces goods used only as intermediate goods (say, coal). Department II produces wage goods used as minimum subsistence (say, corn). Department III produces only luxury goods for capitalists (say, velvets or wampum). Using lower-case letters for "values" and upper-case letters for "prices," he summarizes this model by the relations

$$(8) \quad \begin{aligned} c_1 + v_1 + s_1 &= c_1 + c_2 + c_3 \\ c_2 + v_2 + s_2 &= v_1 + v_2 + v_3 \\ c_3 + v_3 + s_3 &= s_1 + s_2 + s_3 \end{aligned}$$

where of course $s_j = sv_j$ in every case, in accordance with the novel notions of Volume I. The left-hand sides of these relations measure the total costs (including surplus value) of the respective industries, and they are equated to the total revenues of the right-hand sides. But these right-hand sides also embody the assumption that each department has its respective role as intermediate, wage-good, and luxury industry.

The same configuration can be written in its *price* mode as

$$(9) \quad \begin{aligned} C_1 + V_1 + S_1 &= C_1 + C_2 + C_3 \\ C_2 + V_2 + S_2 &= V_1 + V_2 + V_3 \\ C_3 + V_3 + S_3 &= S_1 + S_2 + S_3 \end{aligned}$$

³⁶ It should be pointed out that Marx and his commentators use expressions like prices and values, not for intensive amounts *per unit* of the goods in question, namely P_j and p_j , but rather for the extensive totals of such magnitudes for the industry as a whole, namely for what we moderns would write as $P_j Q_j$ and $p_j Q_j$. As will become clear, no real confusion on this point need arise, particularly since it is often convenient to use our dimensional license and define all $Q_j \equiv 1$ along with total labor $L = 1$, in which case the distinction becomes vacuous.

where $S_j=r(C_j+V_j)$ in every case, in accordance with the Volume III and bourgeois notion of a common interest or profit rate under perfect competition.

To illustrate the “values” relations of (8) we may copy the Bortkiewicz and Tugan-Baranowsky tableau, which is reproduced as Tableau IV in Sweezy [36, 1942, p. 121]. This is based on $s=2/3$ or 66 2/3 percent:

TABLE 3A. VALUES

	Constant Capital	Variable Capital	Surplus Value	Value
I	225	90	60	375
II	100	120	80	300
III	50	90	60	200
	375	300	200	875

To illustrate the “prior” relations of (9) I have modified Bortkiewicz’s tableau, which is shown as Tableau IV in Sweezy, by the trivial change of multiplying all its entries by 15/16, so that total wages or corn are shown as the same numerical amount of needed subsistence, namely as 300 in both cases. Based on a profit rate of $r=1/4$ or 25 percent, we have:

TABLE 3B. PRICES

	Constant Capital	Variable Capital	Surplus Value	Value
I	270	90	90	450
II	120	120	60	300
III	60	90	37½	187½
	450	300	187½	937½

Since only proportions matter, how we determine scale or “normalize” the numbers is of no consequence: any two numbers in the respective tableaux could have been put in bold-face and set equal by convention. Thus, Bortkiewicz equates Department III’s 200, and some Marxians equate the grand totals; some erroneously try to equate *two* numbers, or equally erroneously, to equate certain ratios of two numbers—which, singular cases aside, one cannot do, as Seton [33, 1957] has pointed out.

The equation set (9) is not written down explicitly in Bortkiewicz or Sweezy, but rather in essentially the following equivalent form

$$\begin{aligned}
 & [(C/c)c_1 + (V/v)v_1](1+r) \\
 & \quad = (C/c)(c_1 + c_2 + c_3) \\
 (10) \quad & [(C/c)c_2 + (V/v)v_2](1+r) \\
 & \quad = (V/v)(v_1 + v_2 + v_3) \\
 & [(C/c)c_3 + (V/v)v_3](1+r) \\
 & \quad = (S/s)(s_1 + s_2 + s_3)
 \end{aligned}$$

where

$$\begin{aligned}
 C/c &= C_1/c_1 = C_2/c_2 = C_3/c_3 \\
 (11) \quad V/v &= V_1/v_1 = V_2/v_2 = V_3/v_3 \\
 S/s &= (S_1 + S_2 + S_3)/(s_1 + s_2 + s_3)
 \end{aligned}$$

These last three ratios, which are written as (x, y, z) by Bortkiewicz, can be identified as the respective ratios for the three industries of their unit prices to their unit values, or in my notation as being equal to $(P_1/p_1, P_2/p_2, P_3/p_3) = (y_1, y_2, y_3)$.³⁷ In the above numerical tables, under my normali-

³⁷ In Samuelson [26, 1970], values were denoted by π_j rather than p_j . Greek versus roman letters have often been used to distinguish values from prices.

When this survey was in press I received a paper dealing with a problem usually sidestepped in the literature, namely the handling of varying periods of turnover of capitals (Aleksander Bajt, “A Post Mortem Note on the ‘Transformation Problem,’” *Soviet Studies*, Jan. 1970, 21(3), pp. 371-374). Briefly, I shall indicate here how to handle the case that involves durable or fixed capitals in addition to circulating capital (raw materials) used up completely in each period. Assume m durable capital goods or machines (K_1, \dots, K_m), which are needed in amounts b_{ij} as K_i inputs to produce a unit of Q_j ; let the fraction of equipment used up in such production be d_{ij} . Now add to the n industries m new industries representing gross capital formations of the equipments: thus the old a_{ij} matrix now has j going beyond n to $m+n$. And for $i > n$, the respective new augmented coefficients of a are of the form $(1+r)^{-1}(r+d_{ij})b_{ij}$. Now our augmented relations are $P/W = A_0(r) = a_0(1+r)[I - a(r)(1+r)]^{-1}$, where $a(r)$ is the above-described augmented a matrix. Values are given by $p/w = A_0(0)(1+s)$. As before $A_{0j}'(r) > 0$, and all transformation relations of this paper continue to be valid and do represent genuine improvements over Engel’s editorial repairs to Marx’s fragmentary manuscript of Volume III.

zation $(x, y, z) = (1.2, 1, 15/16)$ rather than $(1.28, 16/15, 1)$ as in Bortkiewicz and Sweezy. Since only the ratios of (x, y, z) are invariants, mere differences in scale are of no significance.

Why should Bortkiewicz have chosen to write C_j and V_j in the roundabout form of $(C_j/c_j)c_j$ and $(V_j/v_j)v_j$? If “values” cancel out of the price calculations, and Samuelson [25, 1970] shows that they indeed do, why introduce them in the first place? Or, at least, why not abandon them immediately and explicitly? Actually, it would have helped to avoid the misunderstandings in the literature of the significance of the transformation process if Bortkiewicz had taken a more straightforward and transparent attack in his transformation of the models of simple reproduction. But there is rhyme and reason to his procedure, as we shall see.

Although this was not brought out by Marx and his pre-1957 commentators, it is absolutely necessary to pick up from somewhere the physical and technical data of the model. Without them no formulation of equilibrium is possible and no valid comparison between the two regimes. Bortkiewicz recognized that if we happen first to have given the values totals for the industries, we can use them to calculate information about the a coefficients, after which the values are indeed abandoned in the computation of prices. Further, in what is important for the understanding of the sense in which Volume I's concepts may be necessary or fruitful for the understanding of the real world and its laws of motions, if first we are given data on the price totals, we must use these data to calculate information about the technical a coefficients, after which prices can be abandoned in the computation of values. Tugan-Baranowsky had already (a few years before 1907) performed [38, 1905] the inverse transformation from prices to values by just such an implicit calculation of the (a_0, a) data; Morishima and Seton [21, 1961] have more recently brought the

subject around full circle. What saves an expression like $(C_j/c_j)c_j$ from being a trivial tautology for c_j is (11)'s assertion that the factor C_j/c_j is the same for all industries, independently of j , in reflection of the fact that there *is* a technology underlying the tableau.

To understand how the (a_0, a) data are “identified” in their relevant aspects, let me rewrite the relations in my notation. In general, we have

$$\begin{aligned} c_j + v_j(1 + s) &= p_1a_{1j}Q_j + p_2a_{2j}Q_j + p_3a_{3j}Q_j \\ &\quad + wL_j(1 + s) = p_jQ_j \\ &\quad (j = 1, 2, 3), \end{aligned} \tag{8'}$$

$$\begin{aligned} w(L_1 + L_2 + L_3) &= w(a_{01}Q_1 + a_{02}Q_2 + a_{03}Q_3) = p_2Q_2, \end{aligned}$$

where (w, p_j) represent the wage and values per unit; and

$$\begin{aligned} (C_j + V_j)(1 + r) &= (P_1a_{1j}Q_j + P_2a_{2j}Q_j + P_3a_{3j}Q_j \\ &\quad + WL_j)(1 + r) = P_jQ_j \\ &\quad (j = 1, 2, 3), \end{aligned} \tag{9'}$$

$$\begin{aligned} W(L_1 + L_2 + L_3) &= W(a_{01}Q_1 + a_{02}Q_2 + a_{03}Q_3) = P_2Q_2, \end{aligned}$$

where (W, P_j) represent the wage and prices per unit.

Six (or $n\{n-1\}$) of our needed twelve (or $n\{n+1\}$) identifications are provided by the assumption that the model is one of “simple” structure, with only one good (coal) being needed as intermediate goods. This tells us that $a_{2j} = 0 = a_{3j}$, for all $j = 1, 2, 3$. Thus, once we have specified the conventional physical units in which $Q_1, Q_2, Q_3, L = \sum L_j$ are to be measured, the remaining six (or $2n$) coefficients $[a_{0j}; a_{1j}]$ can be identified from *either* tableau. Following Marx and not Tugan, let us begin with (8'), the values tableau. For brevity, I skip the general for-

mulation and assume the convention that $Q_1=Q_2=Q_3=L=1$, or rather, to keep the arithmetic simple, I assume $Q_1=Q_2=Q_3=1$ and $L=300$. Then the wage is one, $w=1$, and all values per unit, p_j , are identical with industry total values measured in wage numeraire. Then, for Tables 3A and 3B

$$\begin{aligned}
 a_{11} &= p_1 a_{11} Q_1 / p_1 Q_1 = c_1 / (c_1 + c_2 + c_3) \\
 &= 225 / 375 = 9 / 15 \\
 a_{12} &= [p_1 a_{12} Q_2 / p_2 Q_2] (p_2 / p_1) \\
 &= c_2 / (c_1 + c_2 + c_3) = 100 / 375 = 4 / 15 \\
 a_{13} &= [p_1 a_{13} Q_3 / p_3 Q_3] (p_3 / p_1) \\
 &= c_3 / (c_1 + c_2 + c_3) = 2 / 15 \\
 a_{01} &= [w a_{01} Q_1 / p_1 Q_1] (p_1 / w) \\
 &= L_1 / Q_1 = (90 / 375) (375) = 90 \\
 a_{02} &= [w a_{02} Q_2 / p_2 Q_2] (p_2 / w) = L_2 / Q_2 \\
 &= (120 / 300) (300) = 120 \\
 a_{03} &= [w a_{03} Q_3 / p_3 Q_3] (p_3 / w) = L_3 / Q_3 \\
 &= (90 / 200) (200) = 90
 \end{aligned}$$

For the mathematical reader, I may now quickly write down the Leontief production relations in matrix form

$$\begin{aligned}
 Q &= aQ + \text{consumptions} = aQ + Y \\
 &= [I - a]^{-1} Y = A Y \\
 &= \begin{bmatrix} 1 - \left(\frac{9}{15}\right) & -\left(\frac{4}{15}\right) & -\left(\frac{2}{15}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} \left(\frac{15}{6}\right) & \left(\frac{4}{6}\right) & \left(\frac{2}{6}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
 L &= a_0 Q = [90 \quad 120 \quad 90] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 300 \\
 &= a_0 [I - a]^{-1} Y
 \end{aligned}$$

$$\begin{aligned}
 &= [90 \quad 120 \quad 90] \begin{bmatrix} \left(\frac{15}{6}\right) & \left(\frac{4}{6}\right) & \left(\frac{2}{6}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
 &= A_0(O) Y = [225 \quad 180 \quad 120] \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 300
 \end{aligned}$$

Since only 180 labor units are needed to produce the total subsistence for laborers, *i.e.*, to produce $Q_2=1$, and since we have 300 labor units available, the rate of surplus value is defined by $1+s^*=300/180=1+2/3$.

Our value relations then become, in matrix terms,

$$\begin{aligned}
 p &= w a_0 (1 + s^*) + p a \\
 &= w a_0 (1 + s^*) [I - a]^{-1} = a_0 (1 + s^*) A, \\
 &\hspace{15em} \text{for } w = 1 \\
 &= [90(5/3) \quad 120(5/3) \quad 90(5/3)] A \\
 &= [150 \quad 200 \quad 150] \begin{bmatrix} \left(\frac{15}{6}\right) & \left(\frac{4}{6}\right) & \left(\frac{2}{6}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= [375 \quad 300 \quad 200]
 \end{aligned}$$

which checks with the last column of Table 3A above.

To derive the price relations in matrix form, we must find that unique rate of profit, r^* , which will raise the price of subsistence corn from 180 units to 300, so that the workers will have just enough wage income to buy the $Q_2=1=Y_2$ of subsistence. Prices are defined by

$$\begin{aligned}
 P &= W a_0 (1 + r) + P a (1 + r) \\
 &= W a_0 (1 + r) [I - a(1 + r)]^{-1} \\
 &= a_0 (1 + r) [I - a(1 + r)]^{-1}, \text{ for } W = 1 \\
 &= A_0(r)
 \end{aligned}$$

Rather than solve $A_{02}(r)=300$ for its relevant root, which involves solving a quadratic, we can easily verify that, for r^*

$= 1/4, P_2/W = A_{02}(1/4) = 300$, namely

$$\begin{aligned} \frac{P}{W} &= [90(5/4) \quad 120(5/4) \quad 90(5/4)] \begin{bmatrix} 1 - \left(\frac{9}{15}\right)\left(\frac{5}{9}\right), & -\left(\frac{4}{15}\right)\left(\frac{5}{4}\right), & -\left(\frac{2}{15}\right)\left(\frac{5}{4}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= [450/4 \quad 600/4 \quad 450/4] \begin{bmatrix} 4 & \left(\frac{20}{15}\right) & \left(\frac{10}{15}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= [450 \quad 300 \quad 187\frac{1}{2}] \end{aligned}$$

which checks with the last column of Table 3B.

The analysis of Seton [33, 1957] may be briefly related to that here. Instead of specifying the vector of wage-subsistence $[m_i]$, he adds to the production requirements $[a_{ij}]$ further input requirements needed to feed the workers, $[k_{ij}]$. Thus if every worker consumes $[m_i]$ independently of the industry he works in, Seton's k_{ij} would take the special form $m_i a_{0j}$, and the results given here would be reached by him but by a different path of exposition. The only possible disadvantage of his exposition in that case would be the fact that, every time any m_i changed, he would have to rework the whole of his $I - (a + k)(1 + r)$ matrix calculations; the present conventional Leontief-Walras approach takes advantage of the decomposability of the subsistence requirements from the rest of the relationships. On the other hand, the Seton hypothesis is a genuinely broader empirical one and can handle cases not admissible to the present formulation. A necessary price to pay for this greater generality is that it robs Volume I's surplus-value analysis of its greater algebraic simplicities; instead of the basic relations being linear in $1 + s$, one must in Seton's general case solve an eigenvalue polynomial $|I - a - k(1 + s)| = 0$ that is of as high algebraic degree as is $|I - (a + k) \cdot (1 + r)| = 0$. Morishima and Seton [21,

1961], and Johansen [11, 1961], develop this 1957 model further.

Since my text has already registered its agreements and disagreements with such writers of the 1940s and 1950s as Joan Robinson [25, 1950] and Meek [19, 1956] only a few comments are needed here.³⁸ Winternitz [40, 1948] correctly points out that any luxury goods, such as Department III, which contribute neither directly nor indirectly to the production of subsistence goods, can be ignored in determining the exploitation rate of profit; K. May [18, 1948], for unclear reasons, seems to deny this, but no subsequent writer follows him in this

³⁸ Since the writing of this paper, my old classmate from Chicago days, Professor Martin Bronfenbrenner of Carnegie-Mellon University, has inquired as to the connection between the solution proposed here and the solution to the transformation problem that he had suggested [5, 1965 and 6, 1967]. I pointed out that his 1965 and 1967 formulas, if taken literally, are subject to a defect that goes beyond the defect of the Marx 1894 formulation—namely that the newly proposed competitive prices do not each equal the sum of their costs of production including stipulated profits. Professor Bronfenbrenner writes that in 1971 he would prefer to have his formulas (expressed in *his* notation of p. 210 of the first reference and p. 634 of the second)

$$\begin{aligned} P' &= p_i W_i / (C_i + V_i) \quad \text{replaced by} \\ P' &= [S_i + (p_i - 1)W_i] / (C_i + V_i) \end{aligned}$$

with appropriate modifications being made in his later formulas. Simple manipulation of the new formulas will show that the modified Bronfenbrenner solution is precisely that of Marx's 1894 Volume III, with all the defects of false evaluation of intermediate-good costs referred to by other writers and by my text.

respect. As I have already indicated, writers err in thinking that this decomposability of luxuries is a correction or simplification of Bortkiewicz; I should also add that writers err in believing that such a decomposability property adds to, or subtracts from, the empirical validity of an exploitation model (contrary to vague suggestions among modern commentators on Ricardian constructions).

In a useful survey M. Dobb [9, 1955] points out that he had earlier erred in thinking that the transformation from values to prices would involve a substantive change in resource allocations. Indeed so long as technological coefficients and subsistence-quanta are frozen constants no such change takes place. But Dobb's original instinct was a good one. All classical writers, particularly when discussing inventions, show an awareness that the blueprints of life contain more than one page. As soon as $[a_{0j}, a_{ij}, m_i]$ cease to be uniquely-given constants, but instead have to be selected from two or more options in terms of cost minimization, the discussions reviewed here become overly simple. As one goes from r^* to s^* , and vice versa, the $[a_0, a, m]$ regime may change and the indicated correspondences between them will no longer hold. Now the price regime can no longer be defined by purely algebraic procedures of root extraction; instead, competitive cost minimizations may involve switchings (and reswitchings). Such substitutions cannot happen if cost minimization takes place in terms of values, since relative values, p_i/p_j , are independent of s , as was indicated in the parallel shiftings in Figure 1 and as is clear from the proportionality between values and labor costs. One would expect this to be registered as a defect of the Volume I values scheme, since for planning purposes it requires the richest and poorest societies to use the same production methods. It is difficult, however, to compare the efficiencies of the two regimes until the problem is better posed. Efficiency in terms of what? And for

whom? This much can be said in favor of price-profit as against value-surplus configurations: the former could, and the latter generally could not, serve as the efficient golden-age asymptote of a rationally planned solution to an optimal-control problem of generalized Ramsey and von Neumann type.³⁹ Thus, as soon as goods have some substitutability for each other for workers and non-workers, generally *everybody* can be made better off by a judicious transformation from values to prices. This is an important fact in weighing the merits of the Volume I innovations.

In closing I wish to emphasize that, except inadvertently, I have not discussed any evidence bearing on the empirical usefulness (for the last century or this one) of the exploitation model as throwing light on the laws of motion of the system. In the present paper I have tried only to clarify the logic of that model—in the hope that any evaluation of how it has worked out over time can benefit from an understanding of how it works in every instant of time.

³⁹ While this article was in press C. C. von Weizsäcker and P. A. Samuelson in "A New Labor Theory of Value for Rational Planning Through Use of the Bourgeois Profit Rate" (*Proceedings of the National Academy of Sciences*, June 1971, 68) proved the following theorem: if labor grows at exponential rate $1+g$, and goods are to be priced at their "synchronized labor costs," then the bourgeois pricing formula $A_0(g) = a_0(1+g)[I - a(1+g)]^{-1}$ must be charged by rational planners. More generally, if in addition to population growth at $(1+g)$, invention causes direct labor requirements to decrease like $(1+h)^{-t}$, then pricing goods so that all which does not go to wages goes for "widening" of capital goods will require ratios P_i/P_j to equal $A_{0i}(G)/A_{0j}(G)$, where the profit rate satisfies $1+G = (1+g)(1+h)$.

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